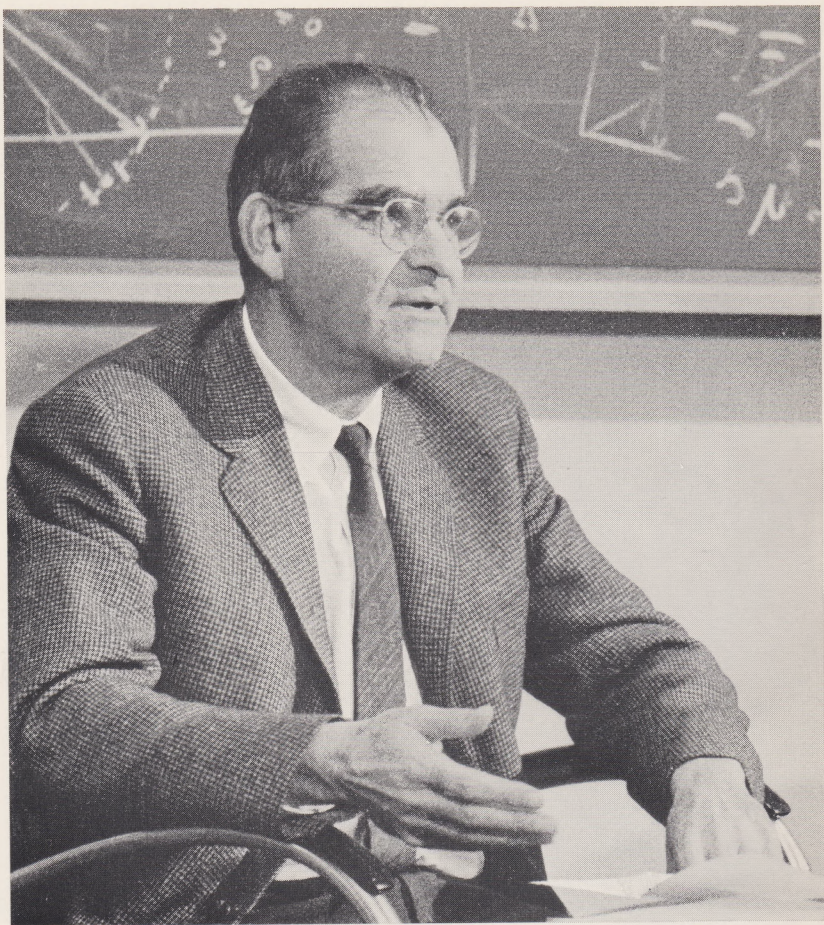


PRELUDES IN THEORETICAL PHYSICS









Victor F. Weisskopf



# PRELUDES IN THEORETICAL PHYSICS

IN HONOR OF V. F. WEISSKOPF

*edited by*

A. DE-SHALIT

*Department of Nuclear Physics,  
The Weizmann Institute of Science, Rehovoth, Israel*

H. FESHBACH

*Department of Physics and Laboratory for Nuclear Science,  
MIT, Cambridge, Mass., USA*

L. VAN HOVE

*Theoretical Study Division, CERN, Geneva, Switzerland*



1966

NORTH-HOLLAND PUBLISHING COMPANY - AMSTERDAM

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PUBLISHERS:

NORTH-HOLLAND PUBLISHING CO. - AMSTERDAM

SOLE DISTRIBUTORS IN THE U.S.A. AND CANADA:

INTERSCIENCE PUBLISHERS, a division of  
JOHN WILEY & SONS, INC. - NEW YORK

PRINTED IN THE NETHERLANDS



## EDITORS' FOREWORD

*Towards the end of 1964 it became known that Viki Weisskopf had decided to go back to MIT after having served as Director General of CERN for five years. Hopes were expressed that Viki might still change his mind, but it became clear that this time his decision was definite.*

*Many of us who have been visiting CERN, and working there, for shorter or longer periods, felt an urge to express our gratitude to Viki for everything he has done during these five years to make CERN such a pleasant and stimulating place. Suggestions of various sorts were brought up, but it seemed to us that this purpose could be best served by collecting together remarks and studies of a special character and dedicating this collection to Viki Weisskopf. Viki has won a special reputation for this insistence on looking at any given problem in physics from a variety of angles, and for his attempts to reduce to bare minimum formal derivations. His "intuitive" way of looking at things has been a source of aesthetic pleasure to everyone who has had the good fortune of working with him. As a matter of fact it is this "philosophy" of his that has given rise to some of the most exciting seminars at CERN and has guided the thinking of many of its scientists. The "preludes" collected in this volume are intended to illustrate some such approaches to a variety of physical problems.*

*The list of Viki's close friends is too long to have been covered in a volume like the present one. We have, therefore, limited our invitations only to those theoretical physicists who visited CERN and spent some time there. We tried to make sure that the list was as complete as possible, but there might have been some omissions, and we express our apologies to them.*

*Finally, Viki's work at CERN would have been impossible without the understanding, encouragement and help of Ellen Weisskopf; we*



wish to take this opportunity to thank her, too, for the warm atmosphere all of us have always found at their home, and for complementing so harmoniously with Viki's contributions to CERN.

A. de-Shalit  
H. Feshbach  
L. Van Hove

## INTRODUCTION

*In 1960 C. J. Bakker was killed in an airplane accident and a new director general of CERN had to be appointed. But also in other respects CERN was then in a state of transition. The construction of the synchrocyclotron and of the big accelerator had been successfully completed; physicists were gradually taking over from engineers and beginning to obtain interesting experimental results. It was important that the original fervour and spirit of cooperation, that had led to the creation of a European centre of high energy physics should be maintained, now that the first building period was over. Important not only to those working at CERN but in a broader sense to all physicists. It has always been the claim of scientists, that they have little difficulty to arrive at international understanding, as long as they are not hampered by the dullness of commercial acumen or the insipidity of diplomatic adroitness. Through CERN they had to prove their point, for this was an organization created by physicists for the pursuit of physics, not by governments for economic purposes or for some vague reasons of prestige.*

*When Viki Weisskopf accepted the appointment this was a great relief even to those who were only indirectly involved, but who knew the man and his background.*

*A few words about this background. Although the years from 1924 to 1935 with their grievous economic depression and the threat and finally the arrival of nazism were in many ways alarming, they will be remembered by theoretical physicists as a happy era. There was the feeling of a great spiritual breakthrough, followed by a surprisingly rich harvest, there was a feeling of belonging to a small and select inner circle headed by a few really outstanding men.*

*Weisskopf, who had worked at Göttingen, Zürich and Copenhagen before moving to the United States, was one of the prominent younger*



members of this group. He worked with Wigner and with Pauli and their power of mathematical penetration left their mark upon him. He knew Ehrenfest well and felt akin to him because of his preference for simple, clear and beautiful formulations. And above all he underwent the influence of Bohr's depth and wisdom.

But while others may wistfully remember those days, it is Weisskopf's unique achievement that he has carried over the devoted idealism and the enthusiasm of his early days into a new world of organized research and large scale experimentation.

Through the work he did at CERN, through the impact of his mature personality, he has had a profound influence on modern physics in Europe.

The present essays, in which we try to capture something of his spirit, is offered to him as a small token of gratitude.

H. B. G. Casimir



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# DIE UNITÄREN DARSTELLUNGEN DER HOMOGENEN LORENTZGRUPPE

MARKUS FIERZ

*E. T. H. Zürich*

*(Received January 17, 1965)*

Wer kann was Kluges, wer was  
Dummes denken, Das nicht die  
Vorwelt schon gedacht?

*(Mephisto)*

Der Gegenstand der folgenden Betrachtungen gehört heute zum klassischen Bestand der mathematischen Physik. Niemand soll darum erwarten, daß ich etwas bieten kann, was nicht andere im wesentlichen schon gesagt hätten [1]. Der Sinn dieser Mitteilung ist darum ein pädagogischer.

Ich möchte eine anschauliche Methode vorführen, die zu den irreduziblen unitären Darstellungen der homogenen Lorentzgruppe führt.

Als Objekt, das wir Lorentztransformationen unterwerfen, wählen wir die Feldstärken  $\mathbf{E}$ ,  $\mathbf{B}$  und den Ausbreitungsvektor  $\mathbf{p}$  einer ebenen, elektromagnetischen Welle in einem festen Punkt des Raumes und der Zeit.

Die Maßeinheiten können stets so gewählt werden, daß

$$|\mathbf{E}| = |\mathbf{B}| = |\mathbf{p}| = p. \quad (1)$$

Diese Normierung ist lorentzinvariant, wie die vierdimensionale Gleichung

$$F_{ik}F^k{}_l = p_i p_l$$

zeigt. Hier entspricht  $F_{ik} = -F_{ki}$  den Feldstärken  $\mathbf{E}$ ,  $\mathbf{B}$  und  $p_k$  ist der zu  $\mathbf{p}$  gehörige lichtartige Vierervektor:

$$p_k p^k = 0.$$

Die drei Vektoren  $\mathbf{E}$ ,  $\mathbf{B}$  und  $\mathbf{p}$  bilden ein orthogonales Tripel im Raum der dreidimensionalen Vektoren. Jede Lorentztransformation läßt ein gegebenes Tripel entweder unverändert, oder sie führt es in



ein anderes über. Und jedes Tripel kann in jedes andere übergeführt werden, da man durch Dopplereffekt  $p$  beliebig ändern kann.

Wir beschreiben die Tripel durch  $p$  und drei Euler'sche Winkel  $\vartheta, \varphi, \psi$ . Dabei soll  $\psi$  die Drehung um die Richtung von  $\mathbf{p}$  beschreiben, die von  $\mathbf{E}$  nach  $\mathbf{B}$  führt.  $\mathbf{p}$  spielt also die Rolle der Figurenachse im symmetrischen Kreisel. Somit ist

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = e^{i\psi} \mathbf{a}(p, \vartheta, \varphi); \quad (2)$$

denn eine Drehung um  $\mathbf{p}$  ist eine solche der  $(\mathbf{E}, \mathbf{B})$ -Ebene in sich.

Bei Lorentztransformationen werden  $\mathbf{p}$  und  $\mathbf{F}$  je unter sich und linear-homogen transformiert.

Im Raume der Tripel ist

$$d\Omega = \frac{d^3 p}{p} d\psi \quad (3)$$

ein invariantes Volumenelement.

Wir betrachten in diesem Raume skalare, in bezug auf  $d\Omega$  quadratintegrierbare Funktionen  $\Phi(p, \vartheta, \varphi, \psi)$ :

$$\int |\Phi|^2 d\Omega = J. \quad (4)$$

Das Integral  $J$  ist invariant, da Lorentztransformationen lediglich eine Substitution der Integrationsvariablen erzeugen. Wir haben somit eine unitäre Darstellung der homogenen Lorentzgruppe im Hilbertraum der  $\Phi$  vor uns.

Setzt man nun

$$\log p = r, \quad \Phi = 1/p \cdot F$$

dann kann man, weil  $\Phi$  quadratintegrierbar ist,  $F$  wie folgt entwickeln:

$$F = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\mu e^{i\mu r} \sum_{j=0}^{\infty} \sum_{l=-j}^{+j} \sum_{m=-j}^{+j} e^{i(l\psi+m\varphi)} \cdot P_{lm}^j(\vartheta) C_{lm}^j(\mu). \quad (5)$$

Die  $P_{lm}^j(\vartheta)$  sind hier die Eigenfunktionen des symmetrischen Kreisels mit dem Impulsmoment  $j$ .  $l$  ist das Impulsmoment um die Figuren-achse,  $m$  dasjenige um die  $z$ -Achse.

Das invariante Integral wird jetzt

$$J = \int_{-\infty}^{+\infty} d\mu \sum_j \sum_l \sum_m |C_{lm}^j(\mu)|^2. \quad (6)$$

Hier ist aber schon

$$J_l(\mu) = \sum_{j \geq |l|} \sum_{|m| \leq j} |C_{lm}^j(\mu)|^2 \quad (7)$$

invariant. Denn zu festem  $\mu$  und  $l$  entsprechen die  $C_{lm}^j(\mu)$  Funktionen  $\Phi$ , die homogen sind in  $p$  vom Grad  $i\mu - 1$ , und homogen sind in  $e^{i\psi}$  vom Grad  $l$ . Da nun  $p$  und  $F$  linear-homogen transformiert werden, bleiben die Homogenitätsgrade  $i\mu - 1$  und  $l$  invariant. Zu festem  $\mu$  und  $l$  bilden die  $C_{lm}^j(\mu)$  also einen invarianten Raum und es ist leicht zu sehen, daß dieser auch irreduzibel ist. Wir haben also durch  $\mu$  und  $l$  charakterisierte, unitäre und irreduzible Darstellungen gewonnen.

Wenn man die Raumspiegelung als weiteres Element der Gruppe hinzunimmt, so hat man neben  $l$  auch  $-l$  zu betrachten, denn die Spiegelung führt  $l$  nach  $-l$ .

Die Darstellungen  $l = 0$  erhält man schon, wenn man  $p$  allein betrachtet.  $\Phi(p)$  ist alsdann eine Funktion auf dem Lichtkegel.

Man kann unsere Tripel als „anschauliche“ Darstellung von Spinoren  $a_\alpha$  ansehen. Es existiert nämlich die folgende, eindeutige Zuordnung:

$$a_\alpha \bar{a}_\beta \rightarrow p, \quad a_\alpha a_\beta \rightarrow E + iB. \quad (8)$$

Einem Tripel sind aber immer zwei Spinoren,  $a_\alpha$  und  $-a_\alpha$  zugeordnet.

Mit Hilfe der Spinoren erkennt man sogleich, daß es Lorentztransformationen gibt, welche ein gegebenes Tripel nicht ändern. Sei nämlich der zugehörige Spinor

$$a_1 = a, \quad a_2 = 0$$

so ändert die unimodulare Transformation

$$a_1 + Ca_2 = a'_1, \quad a_2 = a'_2 \quad (9)$$

den Spinor nicht. Dabei ist  $C$  eine beliebige komplexe Konstante. Im allgemeinen läßt freilich eine Lorentztransformation kein einziges Tripel invariant. Wenn sie jedoch eines invariant läßt, dann auch alle

diejenigen anderen, für welche  $p$  die gleiche Richtung besitzt.

Ist in der Transformation (9)  $C = 2 \operatorname{tg} \alpha$  reell, so kann man die Transformationsmatrix wie folgt aufspalten:

$$\begin{pmatrix} \cos \alpha, & \sin \alpha \\ -\sin \alpha, & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \alpha, & \sin \alpha \\ \sin \alpha, & (1 + \sin^2 \alpha)/\cos \alpha \end{pmatrix} = \begin{pmatrix} 1 & 2 \operatorname{tg} \alpha \\ 0 & 1 \end{pmatrix}. \quad (10)$$

Es sei dem Leser überlassen, den Sinn dieser Darstellung zu ergründen.

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# AN ELEMENTARY DISCUSSION OF POSSIBLE NON-INVARIANCE UNDER $T$ , $CP$ AND $CPT$ IN HYPERON DECAYS \*

T. D. LEE

*Department of Physics, Columbia University, New York, N.Y.*

*(Received April 20, 1965)*

## 1. INTRODUCTION

The question whether the weak interactions are invariant under the time reversal operation  $T$ , or  $CP$  the product of the charge conjugation  $C$  and the space inversion  $P$ , has been raised [1] in the early days when the possibility of non-conservation of parity was being studied. After the discoveries [2, 3] that parity is not conserved, several experiments were performed to test the time reversal invariance in weak interactions. It was found that within the experimental accuracy [4], of about a few % in relative amplitudes, time reversal invariance holds in the  $\beta$ -decay, and to a much lesser accuracy, the same holds [5] for the  $\Lambda^0$  decay. If  $CPT$  invariance [6] is assumed, to the same degree of experimental accuracy  $CP$  is also conserved in these decays.

Recently, Christenson *et al.* observed [7] that the long-lived component of the neutral  $K^0$  meson can decay into  $(\pi^+ + \pi^-)$ , thus suggesting that  $CP$  invariance is violated in the  $K_{2\pi}^0$  decay. The observed non-invariant amplitude is quite small, being only  $\sim 2 \times 10^{-3}$  relative to the corresponding  $CP$  conserving amplitude. If  $CPT$  invariance is assumed, then the same experiment implies that time reversal invariance is also violated.

The experiments which established parity non-conservation usually consist of directly observing a right-left asymmetric effect from an, otherwise, initially right-left symmetric state. The conclusions that space inversion symmetry is violated in these experiments can be reached without any theoretical assumptions. The same is also true for the violation of charge conjugation symmetry. It is important to note

\* This research was supported in part by the U.S. Atomic Energy Commission.

that in all these weak interaction experiments, which pertain to testing the time reversal invariance or non-invariance, not a single one consists of comparing a reaction with its time reversal process. The relations between these experimental observations and time reversal symmetry are obtained through indirect theoretical reasoning, and some of these conclusions are valid only under additional assumptions such as *CPT* invariance. Similar criticism applies also to many of the existing tests of *CP* invariance.

It seems, therefore, desirable to review the underlying theoretical arguments of some of these tests, and to separate out the various implications of different symmetry requirements. With this motive, we will analyze in this note the simple example of the decay of a spin  $\frac{1}{2}$  hyperon, say,

$$\Lambda^0 \rightarrow N + \pi \quad (1)$$

where  $N$  stands for either  $p$  or  $n$  and  $\pi$  represents the corresponding  $\pi^-$  or  $\pi^0$ . The consequences of possible non-invariance under  $T$ ,  $CP$  and *CPT* are derived in Section II. As is well known, the time reversal invariance in the hyperon decay means [8] that the relative phase between the final  $s_{\frac{1}{2}}$  and  $p_{\frac{1}{2}}$  amplitudes is determined by the corresponding strong interaction phase shifts. In Section III, the same result is obtained by an alternative proof which is based only on the reciprocity relation between different reaction rates, without the explicit use of the time reversal operator [9]  $T$ . A simple example is given in Section IV which illustrates the difference between the consequences of time reversal invariance in quantum mechanics and that in classical mechanics, and which emphasizes again that the present tests of time reversal invariance concern only the reciprocity relations between various differential cross-sections, rather than the detailed time reversal operation  $T$ .

Throughout these discussions we assume that the amplitude of reaction (1) can be represented by the corresponding matrix element of a Hermitian operator [10]  $H_{\text{weak}}$ . The validity of the local field theory, or *CPT* invariance, is not assumed.

## 2. $\Lambda^0$ DECAY

In the decay of  $\Lambda^0$ , the final  $(N + \pi)$  system can be in either a  $s_{\frac{1}{2}}$  or a

$p_{\frac{1}{2}}$  spin-orbital state. Let these two transition amplitudes be denoted by  $A_s(I)$  and  $A_p(I)$  where  $I = \frac{1}{2}$ , or  $\frac{3}{2}$ , is the total iso-spin of the final state. The relative phase  $\phi(I)$  is given by

$$\frac{A_s(I)}{A_p(I)} = \left| \frac{A_s(I)}{A_p(I)} \right| e^{i\phi(I)}. \quad (2)$$

Similarly, in the decay of the anti-lambda,

$$\bar{\Lambda}^0 \rightarrow \bar{N} + \pi, \quad (3)$$

the corresponding  $s_{\frac{1}{2}}$  and  $p_{\frac{1}{2}}$  transition amplitudes are  $\bar{A}_s(I)$  and  $\bar{A}_p(I)$ , and their relative phase  $\bar{\phi}(I)$  is given by

$$\frac{\bar{A}_s(I)}{\bar{A}_p(I)} = \left| \frac{\bar{A}_s(I)}{\bar{A}_p(I)} \right| e^{i\bar{\phi}(I)}. \quad (4)$$

The following theorem states the separate consequences of the invariance requirements under  $T$ ,  $CP$  and  $CPT$  for the  $\Lambda^0$  and  $\bar{\Lambda}^0$  decays. Throughout the present paper, we neglect the effects of electromagnetic interactions and assume that the strong interaction is separately invariant under  $T$ ,  $C$  and  $P$ .

#### Theorem

1. If  $T$  invariance holds then, independent of  $CP$  invariance,

$$\phi(I) = \begin{cases} \delta_s(I) - \delta_p(I), & \text{or} \\ \delta_s(I) - \delta_p(I) - \pi \end{cases} \quad (5)$$

and

$$\bar{\phi}(I) = \begin{cases} \delta_s(I) - \delta_p(I), & \text{or} \\ \delta_s(I) - \delta_p(I) - \pi \end{cases} \quad (6)$$

where  $\delta_s(I)$  and  $\delta_p(I)$  are, respectively, the  $s_{\frac{1}{2}}$  and  $p_{\frac{1}{2}}$  phase shifts due to the strong interactions of the  $(N + \pi)$  system with a total iso-spin  $I$ .

2. If  $CP$  invariance holds then, independent of  $T$  invariance,

$$A_s(I) = -\bar{A}_s(I) \quad (7)$$

$$A_p(I) = +\bar{A}_p(I) \quad (8)$$

and, consequently,

$$\phi(I) = \bar{\phi}(I) + \pi. \quad (9)$$



For convenience, we chose the anti-particle states  $\bar{\Lambda}^0$  and  $\bar{N}$  to be identically related to their respective particle states  $\Lambda^0$  and  $N$  through the  $CP$  operation.

3. If  $CPT$  invariance holds then, independent of either  $T$  invariance or  $CP$  invariance,

$$|A_s(I)| = |\bar{A}_s(I)|, \quad (10)$$

$$|A_p(I)| = |\bar{A}_p(I)| \quad (11)$$

and

$$\frac{1}{2}[\phi(I) + \bar{\phi}(I)] = \begin{cases} [\delta_s(I) - \delta_p(I)] + \frac{1}{2}\pi, & \text{or} \\ [\delta_s(I) - \delta_p(I)] - \frac{1}{2}\pi. \end{cases} \quad (12)$$

Some of these results, e.g. Eqs. (5) and (6), are well known and have already been proved in the literature [8]. For pedagogical reasons, a formal proof of this theorem is given below.

*Proof.* We consider the rest system of  $\Lambda^0$ . Let  $|(N\pi)_{I,s}^{\text{st}}\rangle$  and  $|(N\pi)_{I,p}^{\text{st}}\rangle$  be, respectively, the stationary wave eigen-states of the strongly interacting  $(N+\pi)$  system in the  $s_{\frac{1}{2}}$  and  $p_{\frac{1}{2}}$  orbits and with a total iso-spin  $I = \frac{1}{2}$  or  $\frac{3}{2}$ . The corresponding incoming wave states  $|(N\pi)_{I,s}^{\text{in}}\rangle$  and  $|(N\pi)_{I,p}^{\text{in}}\rangle$  are related to these stationary states by

$$|(N\pi)_{I,l}^{\text{in}}\rangle = e^{-i\delta_l(I)} |(N\pi)_{I,l}^{\text{st}}\rangle \quad (13)$$

where  $l = s$  or  $p$ . The transition amplitude  $A_l(I)$  is given by

$$A_l(I) = \langle (N\pi)_{I,l}^{\text{in}} | H_{\text{weak}} | \Lambda^0 \rangle \quad (14)$$

$$= e^{+i\delta_l(I)} \langle (N\pi)_{I,l}^{\text{st}} | H_{\text{weak}} | \Lambda^0 \rangle \quad (15)$$

where  $|\Lambda^0\rangle$  is the physical  $\Lambda^0$  state. The time reversal operator  $T$  is represented [11] by the joint operation of a complex conjugation times a unitary operator  $U_T$ . Since the strong interaction is invariant under  $T$ , all its stationary eigen-states  $|j, m\rangle$  which have zero total momentum can be chosen to transform under  $T$  as

$$T|j, m\rangle = U_T|j, m\rangle^* = (-1)^m |j, -m\rangle, \quad (16)$$

where  $j$  is the total angular momentum quantum number and  $m$  its  $z$ -component. Both  $|\Lambda^0\rangle$  and  $|(N\pi)_{I,l}^{\text{st}}\rangle$  satisfy Eq. (16). If  $H_{\text{weak}}$  is invariant under the time reversal operation, then

$$TH_{\text{weak}}T^{-1} = U_TH_{\text{weak}}^*U_T^\dagger = H_{\text{weak}}, \quad (17)$$

and, as a consequence,  $\langle (\bar{N}\pi)_{I,l}^{\text{st}} | H_{\text{weak}} | \Lambda^0 \rangle$  are real. Thus, Eq. (5) can be obtained by using Eq. (15).

For the decay of  $\bar{\Lambda}^0$ , we may denote the corresponding incoming wave eigen-state of the strongly interacting  $(\bar{N} + \pi)$  system by  $|(\bar{N}\pi)_{I,l}^{\text{in}}\rangle$ . Since the strong interaction is invariant under  $C$ , Eq. (13) implies that the  $|(\bar{N}\pi)_{I,l}^{\text{in}}\rangle$  state is also related to the stationary state  $|(\bar{N}\pi)_{I,l}^{\text{st}}\rangle$  by

$$|(\bar{N}\pi)_{I,l}^{\text{in}}\rangle = e^{-i\delta_l(I)} |(\bar{N}\pi)_{I,l}^{\text{st}}\rangle. \quad (18)$$

Equation (6) can be derived by using the relation

$$\bar{A}_l(I) = \langle (\bar{N}\pi)_{I,l}^{\text{in}} | H_{\text{weak}} | \bar{\Lambda}^0 \rangle. \quad (19)$$

To establish the consequences of  $CP$  invariance, we may choose

$$|\bar{\Lambda}^0\rangle = CP|\Lambda^0\rangle, \quad (20)$$

$$|(\bar{N}\pi)_{I,p}^\alpha\rangle = +CP|(\bar{N}\pi)_{I,p}^\alpha\rangle \quad (21)$$

and

$$|(\bar{N}\pi)_{I,s}^\alpha\rangle = -CP|(\bar{N}\pi)_{I,s}^\alpha\rangle \quad (22)$$

where  $\alpha$  = stationary or incoming. Equations (7)–(9) are the direct consequences of the assumption that  $H_{\text{weak}}$  is invariant under  $CP$ , i.e.

$$CPH_{\text{weak}}P^{-1}C^{-1} = H_{\text{weak}}. \quad (23)$$

If  $H_{\text{weak}}$  is invariant under  $CPT$ , then

$$CPTH_{\text{weak}}T^{-1}P^{-1}C^{-1} = H_{\text{weak}}. \quad (24)$$

Equations (10), (11) and (12) follow immediately by using Eqs (14)–(16) and (18)–(22). A special consequence of Eqs (10) and (11) is that  $CPT$  invariance implies [1] the equality of life time between  $\Lambda^0$  and  $\bar{\Lambda}^0$ .

We note that Eqs (5) and (6) are consequences of Eqs (7)–(12) [i.e.  $T$  invariance is a consequence of  $CP$  invariance and  $CPT$  invariance], Eqs (7)–(9) are consequences of Eqs (5), (6) and (10)–(12) [i.e.  $T$  invariance and  $CPT$  invariance imply  $CP$  invariance], and that Eqs (10)–(12) are consequences of Eqs (5)–(9) [i.e.  $T$  invariance and  $CP$  invariance imply  $CPT$  invariances].

The absolute magnitudes and the relative phases of these transition amplitudes can be directly measured by studying the decay rates and

the spin directions [5, 12] for reactions (1) and (3). As shown in [12], if in reaction (1) the initial  $\Lambda^0$  is at rest and is completely polarized along the unit vector  $\hat{S}_\Lambda$ , then at any given momentum direction  $\hat{k}$  the final nucleon is also completely polarized, and its spin direction  $\hat{S}_N$  (measured in its own rest system) is given by

$$\hat{S}_N = [1 - \alpha \cos \theta]^{-1} [(-\alpha + \cos \theta)\hat{k} + \beta(\hat{k} \times \hat{S}_\Lambda) + \gamma(\hat{k} \times \hat{S}_\Lambda) \times \hat{k}] \quad (25)$$

where  $\hat{k}$  and  $\hat{S}_N$  are unit vectors,  $\cos \theta = \hat{k} \cdot \hat{S}_\Lambda$ ,

$$\alpha = 2 \operatorname{Re} (A_s^* A_p) / [|A_s|^2 + |A_p|^2], \quad (26)$$

$$\beta = -2 \operatorname{Im} (A_s^* A_p) / [|A_s|^2 + |A_p|^2], \quad (27)$$

and

$$\gamma = [|A_s|^2 - |A_p|^2] / [|A_s|^2 + |A_p|^2]. \quad (28)$$

The  $A_s$  and  $A_p$  are, respectively, the corresponding  $s_{\frac{1}{2}}$  and  $p_{\frac{1}{2}}$  amplitudes which are linearly related to the  $A_s(I)$  and  $A_p(I)$  by using the appropriate Clebsch-Gordon coefficients depending on whether the nucleon is p or n. The measurements of the rates and the parameters  $\alpha$ ,  $\beta$ ,  $\gamma$  for the decays of  $\Lambda^0$  and  $\bar{\Lambda}^0$  give direct tests of  $T$ , or  $CP$ , or  $CPT$ , invariance in these reactions.

Among these tests, the ones for  $CP$  invariance consist of directly comparing two  $CP$  conjugate processes; their physical implications are, therefore, self-evident. The tests for  $T$  invariance and  $CPT$  invariance are less intuitively obvious. For this reason, an alternative proof of Eqs (5), (6) and (10)–(12), based on reciprocity relations, will be given in the next section.

### 3. RECIPROCITY

In order to make clear the consequence of time reversal symmetry, reaction (1)

$$\Lambda^0 \rightarrow N + \pi$$

should be considered together with its reversed process

$$N + \pi \rightarrow \Lambda^0. \quad (29)$$

Let  $\langle \hat{k}, \hat{S}_N | M | \hat{S}_\Lambda \rangle$  and  $\langle \hat{S}'_\Lambda | M | \hat{k}', \hat{S}'_N \rangle$  be, respectively, the transition matrix elements of reactions (1) and (29) where  $\hat{S}_\Lambda$ ,  $\hat{S}_N$ ,  $\hat{k}$  are the same unit vectors as those used in Eq. (25) and  $\hat{S}'_\Lambda$ ,  $\hat{S}'_N$ ,  $\hat{k}'$  are the corresponding unit vectors for reaction (29).

If time reversal symmetry holds, then reaction *rates* of (1) and (29) are related by the reciprocity relation which states that for arbitrary directions  $\hat{k}$ ,  $\hat{S}_N$  and  $\hat{S}_A$ ,

$$|\langle \hat{k}, \hat{S}_N | M | \hat{S}_A \rangle| = |\langle -\hat{S}_A | M | -\hat{k}, -\hat{S}_N \rangle|. \quad (30)$$

This reciprocity relation deals directly with observations. It is important to note that the usual  $T$  invariance implies not only the transition probabilities but also the transition amplitudes satisfying the reciprocity relation [13, 14]. In this section, we will show that the previously proved consequences of  $T$  invariance can be derived by using *only the reciprocity relations between the relevant reaction rates*.

The reciprocity relation, Eq. (30), holds for the  $(N+\pi)$  system in any isotopic spin state  $I$ . To first order in  $H_{\text{weak}}$  (but all orders in the strong interaction), the transition matrix elements for reactions (1) and (29) are given, respectively, by

$$\langle \hat{k}, \hat{S}_N | M | \hat{S}_A \rangle = \langle (\hat{k}, \hat{S}_N)_I^{\text{in}} | H_{\text{weak}} | \hat{S}_A \rangle \quad (31)$$

and

$$\langle \hat{S}'_A | M | \hat{k}', \hat{S}'_N \rangle = \langle \hat{S}'_A | H_{\text{weak}} | (\hat{k}', \hat{S}'_N)_I^{\text{out}} \rangle \quad (32)$$

where  $|\hat{S}_A\rangle$  and  $|\hat{S}'_A\rangle$  are the same physical  $|\Lambda^0\rangle$  state used in Eq. (14), but with the  $\Lambda^0$  spin polarized along  $\hat{S}_A$  and  $\hat{S}'_A$  respectively. The  $|(\hat{k}, \hat{S}_N)_I^{\text{out}}\rangle$ , (or  $|(\hat{k}, \hat{S}_N)_I^{\text{in}}\rangle$ ) state is the outgoing (or incoming) eigenstate of the strong interaction for the  $(N+\pi)$  system and, in the coordinate representation, it has an asymptotic form that consists of a plane wave with momentum  $k$  and spin  $\hat{S}_N$  plus the appropriate outgoing (or incoming) waves. These eigenstates can be expanded in terms of the spherical waves used in the previous section. For example, in the non-relativistic limit, the explicit asymptotic forms of these expansions in the coordinate representation are given by

$$\begin{aligned} \langle r | (\hat{k}, \hat{S}_N)_I^{\text{out(in)}} \rangle &\rightarrow e^{\pm i\delta_s(I)} U_N(kr)^{-1} \sin[kr + \delta_s(I)] + \\ &+ ie^{\pm i\delta_p(I)} (\boldsymbol{\sigma} \cdot \mathbf{r})(\boldsymbol{\sigma} \cdot \hat{k}) U_N(kr^2)^{-1} \sin[kr - \frac{1}{2}\pi + \delta_p(I)] + \dots \end{aligned} \quad (33)$$

as the relative distance  $r = |\mathbf{r}| \rightarrow \infty$ , where the  $+$  signs in the exponents are for the outgoing state and the  $-$  signs are for the incoming state. The components of the vector  $\boldsymbol{\sigma}$  are the usual Pauli



matrices, and  $U_N$  is a Pauli spinor which satisfies

$$(\boldsymbol{\sigma} \cdot \hat{S}_N)U_N = U_N. \quad (34)$$

Equation (31) and the rotational symmetry property of  $H_{\text{weak}}$  allow us to write

$$\langle \hat{k}, \hat{S}_N | M | \hat{S}_A \rangle = U_N^\dagger [a_s(I)e^{i\delta_s(I)} + a_p(I)e^{i\delta_p(I)}(\boldsymbol{\sigma} \cdot \hat{k})] U_A \quad (35)$$

where  $\dagger$  denotes the Hermitian conjugation and the spinor  $U_A$  satisfies

$$(\boldsymbol{\sigma} \cdot \hat{S}_A)U_A = U_A. \quad (36)$$

The  $a_s(I)$  and  $a_p(I)$  are related to the  $A_s(I)$  and  $A_p(I)$  of the previous section by

$$A_l(I) = a_l(I)e^{i\delta_l(I)} \quad (37)$$

where  $l = s$  or  $p$ . By using the assumed Hermiticity property of  $H_{\text{weak}}$ , the explicit form of  $|\langle \hat{k}, \hat{S}_N \rangle_I^{\text{out}}\rangle$  and Eq. (32), we find that the corresponding matrix element for reaction (29) is given by

$$\langle \hat{S}'_A | M | \hat{k}', \hat{S}'_N \rangle = U_A'^\dagger [a_s^*(I)e^{i\delta_s(I)} + a_p^*(I)e^{i\delta_p(I)}(\boldsymbol{\sigma} \cdot \hat{k}')] U_N' \quad (38)$$

where  $U_N'$  and  $U_A'$  are the spinors whose spin directions are  $\hat{S}'_N$  and  $\hat{S}'_A$  respectively. Substituting Eqs (35) and (38) into Eq. (30), we find that if the reciprocity relation is satisfied then

$$\frac{a_p^*(I)}{a_p(I)} = \frac{a_s^*(I)}{a_s(I)} \quad (39)$$

which gives Eq. (5).

Equations (35) and (38) also determine directly the transition amplitude of the resonant scattering

$$N + \pi \rightarrow A^0 \rightarrow N + \pi. \quad (40)$$

At the resonant energy, the amplitude of (40) is proportional to

$$\begin{aligned} & \langle \hat{k}, \hat{S}_N | M | \hat{k}', \hat{S}'_N \rangle \\ &= U_N^\dagger [a_s e^{i\delta_s} + a_p e^{i\delta_p}(\boldsymbol{\sigma} \cdot \hat{k})] [a_s^* e^{i\delta_s} + a_p^* e^{i\delta_p}(\boldsymbol{\sigma} \cdot \hat{k}')] U_N' \end{aligned} \quad (41)$$

where  $\hat{S}'_N$  and  $\hat{k}'$  are, respectively, the spin and momentum directions of the initial nucleon and  $\hat{S}_N$ ,  $\hat{k}$  are that of the final nucleon. In Eq. (41), we suppress the explicit  $I$ -dependence in  $a_s$  and  $a_p$ . The reciprocity relation between the transition probabilities for the resonant scattering

is given by

$$|\langle \hat{k}, \hat{S}_N | M | \hat{k}', \hat{S}'_N \rangle| = |\langle -\hat{k}', -\hat{S}'_N | M | -\hat{k}, -\hat{S}_N \rangle|, \quad (42)$$

which can also be used to derive Eq. (39), or Eq. (5). For example, let us consider the simple case of a backward resonant scattering, i.e.

$$\hat{k} = -\hat{k}'. \quad (43)$$

Equation (41) becomes simply

$$\langle \hat{k}, \hat{S}_N | M | -\hat{k}, \hat{S}'_N \rangle = U_N^\dagger [C + D(\boldsymbol{\sigma} \cdot \hat{k})] U'_N, \quad (44)$$

where

$$C = |a_s|^2 e^{2i\delta_s} - |a_p|^2 e^{2i\delta_p} \quad (45)$$

and

$$D = (a_s^* a_p - a_p^* a_s) e^{i(\delta_s + \delta_p)}. \quad (46)$$

The transition probability for the resonant scattering from, say,  $\hat{S}'_N = \hat{k}$  to  $\hat{S}_N = \hat{k}$  is proportional to  $(C+D)$ , and the corresponding probability for the reversed process from  $\hat{S}'_N = -\hat{k}$  to  $\hat{S}_N = -\hat{k}$  is proportional to  $(C-D)$ . Thus, if the reciprocity relation, Eq. (42), holds,  $D$  must be zero, which gives another derivation of Eq. (5).

In the same way, by comparing the reaction rates between (3) and

$$\bar{N} + \pi \rightarrow \bar{\Lambda}^0, \quad (47)$$

we can derive Eq. (6) without explicitly using the anti-unitary operator  $T$ ; similarly, Eqs (10)–(12) can be derived by comparing the reaction rates between (1) and (47).

In this simple case of  $\Lambda^0$  and  $\bar{\Lambda}^0$  decays, we have shown that all the consequences of  $T$  invariance and  $CPT$  invariance can be derived by using only reciprocity relations between the various differential cross-sections. The same can also be established for other presently proposed tests of  $T$  invariance and  $CPT$  invariance in weak interactions [15].

#### 4. DISCUSSIONS

In the decay of  $\Lambda^0 \rightarrow N + \pi$ , if the initial  $\Lambda^0$  is completely polarized along  $\hat{S}_\Lambda$ , then at any given momentum direction  $\hat{k}$  the final nucleon must also be completely polarized along  $\hat{S}_N$  which is given by Eq. (25). Let us now consider the reversed reaction  $N + \pi \rightarrow \Lambda^0$ , where the

initial polarization direction  $\hat{S}'_N$  and the incident momentum direction  $\hat{k}'$  are given by

$$\hat{S}'_N = -\hat{S}_N \quad \text{and} \quad \hat{k}' = -\hat{k}. \quad (48)$$

We note that had the system obeyed *classical* mechanics, then time reversal invariance would imply that the final  $\Lambda^0$  in the reversed reaction must be completely polarized along the reversed direction  $\hat{S}'_A$  where

$$\hat{S}'_A = -\hat{S}_A. \quad (49)$$

For the quantum mechanical system, while the final  $\Lambda^0$  does remain completely polarized, its direction  $\hat{S}'_A$  is, in general, *different* from  $-\hat{S}_A$ .

To demonstrate this, we may consider the special case

$$A_s(I) = -A_p(I) \quad (50)$$

and neglect  $\delta_s(I)$  and  $\delta_p(I)$ . Eq. (5) is, then, satisfied. The final nucleon in the decay  $\Lambda^0 \rightarrow N + \pi$  is now always polarized along  $\hat{S}_N = -\hat{k}$  while the final  $\Lambda^0$  in the reversed reaction  $N + \pi \rightarrow \Lambda^0$  is always polarized along  $\hat{S}'_A = -\hat{k}'$  which can be very different from  $-\hat{S}_A$ .

The time reversal operator  $T$  in quantum mechanics relates the solution of the Schrodinger Equation  $\psi(t)$  at a time  $t$  with that at  $-t$ . In the decay  $\Lambda^0 \rightarrow N + \pi^-$ , the final state  $\psi(t = \infty)$  is a coherent mixture of  $s_{\frac{1}{2}}$  and  $p_{\frac{1}{2}}$  waves which, of course, can also be expanded as another coherent mixture of  $|(\hat{k}, \hat{S}_N)_I^{\text{in}}\rangle$  states. Under  $T$ , the state  $|(\hat{k}, \hat{S}_N)_I^{\text{in}}\rangle$  becomes  $|(-\hat{k}, -\hat{S}_N)_I^{\text{out}}\rangle$ , and  $T\psi(t = \infty)$  becomes a corresponding *coherent mixture* of  $|(-\hat{k}, -\hat{S}_N)_I^{\text{out}}\rangle$  states which is, obviously, very different from a single  $|(-\hat{k}, -\hat{S}_N)_I^{\text{out}}\rangle$  state. However, as stated in Eq. (48), it is precisely this single  $|(-\hat{k}, -\hat{S}_N)_I^{\text{out}}\rangle$  state that is being used as the initial state in the reversed reaction  $N + \pi \rightarrow \Lambda^0$ . On the other hand, the reciprocity relation, Eq. (30), does equate the transition probabilities between the  $\Lambda^0$  decay and the reversed reaction whose initial state is given by Eq. (48).

The above simple example merely illustrates once again these elementary aspects of quantum mechanics. It also illustrates that while the mathematical operation of the anti-unitary operator  $T$  deals with the symmetry between the solution  $\psi$  at a time  $t$  and that



at  $-t$ , the direct experimental test of such symmetry properties usually does not go beyond the reciprocity relations between various reaction rates. In connection with the recent observation [7] of Christenson *et al.*, while we can at least envisage theoretically the possibility that, in some distant future, it may become possible to test directly the relevant reciprocity relations [16], it seems virtually impossible to ever construct the desired coherent time reversed state  $T\psi(t = \infty)$  for a direct testing of the symmetry (or, violation of symmetry) properties of the time reversal operation.

I wish to thank Professors G. Feinberg, R. Serber and G. C. Wick for several enjoyable discussions.

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- 9) The usual  $T$  invariance preserves the magnitude  $|\langle\psi|\phi\rangle| = |T\psi|T\phi\rangle|$  for all  $\psi$  and  $\phi$  in the Hilbert space, while in this note the reciprocity relation between various reaction rates refers specifically only to those  $\psi$  and  $\phi$  which represent asymptotically the appropriate initial and final systems in which every particle has a definite momentum and a definite spin. Thus, by itself, the  $T$  invariance appears to be a stronger mathematical condition.
- 10) If, as proposed by Lee and Wolfenstein (Phys. Rev. to be published), the coupling constant of the  $T$  non-invariant interaction, called  $H_F$ , is  $\sim 10^3$  times that of the usual  $T$  invariant weak interaction, called  $H_G$ , then the relevant operator

$H_{\text{weak}}$  would be the sum of  $H_G$  plus the second order term due to  $H_F H_G$ ; otherwise,  $H_{\text{weak}} = H_G + H_F$ .

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- 15) The mass equality of  $p$  and  $\bar{p}$ , which is valid to all orders in  $H_{\text{weak}}$  if  $CPT$  invariance holds, can be derived by using the reciprocity relation between, say,  $\gamma + p \rightarrow \gamma + p$  and its  $CPT$  conjugate process  $\gamma + \bar{p} \rightarrow \gamma + \bar{p}$ . The same reciprocity relation leads also to the well known identities between the electromagnetic properties of  $p$  and  $\bar{p}$ .
- 16) Such tests may become feasible in the immediate future if the coupling constant  $F$  of the  $T$  non-invariant interaction  $H_F$  turns out to be  $\sim 10^3$  times the Fermi coupling constant  $G$  of the usual weak interaction. [See reference [10]]. In such a case, all strong reactions can violate the reciprocity relation by a fractional difference  $\sim (10^{-2} - 10^{-3})$  between the relevant reaction rates. The current experimental accuracy of reciprocity relation in strong interactions is  $\sim 2\%$  as determined by L. Rosen and J. E. Brolley, Jr., Phys. Rev. Letters **2** (1959) 98 for the reactions  $p + t \rightleftharpoons d + d$ . This accuracy is compatible with  $F \sim 10^3 G$ , but it also implies that  $F$  cannot be much bigger than  $10^3 G$ . [Cf., however, the discussion by J. Prentki and M. Veltman, Physics Letters **15** (1965) 88 in which a different view is attempted.]

# BORN APPROXIMATION AND DISPERSION RELATIONS FOR SINGULAR POTENTIALS

A. MARTIN

*CERN, Geneva*

*(Received April 20, 1965)*

The first thing I learnt in physics was scattering theory and it is, maybe, the only thing I know at present. My first teachers were Viki Weisskopf and Léon Van Hove at Les Houches in 1951. So I thought that it might be appropriate to honour Professor Weisskopf with the presentation of some recent developments in this field.

Recently, much interest has been devoted to singular repulsive potentials [1]. Singular because the interactions between elementary particles may be singular, repulsive because we cannot treat the attractive case. It is desirable to show that these interactions are as honest as possible in all what concerns physics, and one thing one would like, in particular, is that their forward scattering amplitude should satisfy dispersion relations. This is not completely obvious from the existing work in this field because partial wave amplitudes have essential singularities at infinity.

We shall start, in section 1, by showing that at least for positive energies ( $k^2 < 0$ ) and negative energies ( $k^2 > 0$ ) the partial wave amplitudes exist in the singular case and can be bounded by a known integral.

In section 2 it will be shown that the forward scattering amplitude exists and can be bounded both for  $k^2 > 0$  and  $k^2 < 0$ .

Section 3 will show how one can “continue” dispersion relations from the non-singular case to the singular case.

## 1. EXISTENCE AND UPPER BOUNDS ON PARTIAL WAVES

In all that follows, we shall restrict ourselves to the case of a *purely* repulsive potential  $V(r)$ , with finite range  $R$ . We shall kill the singularity at the origin by putting in a damping factor:

$$V_\epsilon(r) = e^{-\epsilon/r} V(r). \quad (1)$$



Then the Schrödinger equation reads for the  $l^{\text{th}}$  partial wave

$$u_{el}'' = \left[ \frac{l(l+1)}{r^2} + V_\varepsilon(r) - k^2 \right] u_{el} = V_{\text{eff}}(\varepsilon, r) u_{el} \quad (2)$$

$(k^2 > 0).$

In the physical region we normalize  $u_l$  to be

$$u_{el} = u_{0l} + \text{tg } \delta_l v_{0l} \quad \text{for } r \geq R \quad (3)$$

where  $u_{0l}$  and  $v_{0l}$  are the free solutions behaving like  $\sin(kr - \frac{1}{2}l\pi)$  and  $\cos(kr - \frac{1}{2}l\pi)$ . So that

$$\text{tg } \delta(l, \varepsilon) = -\frac{1}{k} \int_0^R u_{0l} V_\varepsilon(r) u_{el} dr. \quad (4)$$

The crucial remark then is that for  $l > kR$ ,  $V_{\text{eff}}(\varepsilon, r)$  has a constant positive sign and therefore  $u_{el}(r)$  has necessarily a constant sign in the interaction region.

$u_{0l}(R)$  and  $v_{0l}(R)$  are known Bessel functions and they are both positive as long as  $R$  is to the left of the turning point of the Bessel equation, which is the case for  $l > kR$ . Thus, let us show that it follows that  $u_{el}(r)$  is positive for  $0 \leq r \leq R$ . Indeed, if it were negative,  $\text{tg } \delta(l, \varepsilon)$  would be positive from equation (4) and one would get from Eq. (3) that  $u_{el}(R)$  is positive, so  $u_{el}(r)$  is positive and  $\text{tg } \delta(l, \varepsilon)$  is negative.

Hence we have

$$0 < u_{el}(R) < u_{0l}(R). \quad (5)$$

Then since  $V_{\text{eff}}(\varepsilon, r)$  is always larger than  $(l(l+1))/r^2 - k^2$  we notice that  $u_{el}(r)$  is more "curved" than  $u_{0l}(r)$ ; it follows that  $u_{el}(r)$  and  $u_{0l}(r)$  will never intersect in  $0 \leq r \leq R$ . Therefore

$$0 \leq u_{el}(r) < u_{0l}(R) \quad (6)$$

and

$$0 > \text{tg } \delta(l, \varepsilon) > \text{tg } \delta_{\text{Born}}(l, \varepsilon) = -\frac{1}{k} \int_0^R (u_{0l}(r))^2 V_\varepsilon(r) dr \quad (7)$$

and

$$0 > \text{tg } \delta(l, \varepsilon) > -\frac{1}{k} \int_0^R (u_{0l}(r))^2 V(r) dr.$$

Let us show now that  $\operatorname{tg} \delta(l, \varepsilon)$  has a limit for  $\varepsilon \rightarrow 0$ . Indeed using the standard Wronskian technique (as can be found for instance in Blatt and Weisskopf [2]) to compare the solutions  $u_{el}$ ,  $u_{e'l}$ , we can show that

$$\frac{d}{d\varepsilon} [\operatorname{tg} \delta(l, \varepsilon)] = -\frac{1}{k} \int_0^R [u_{el}(r)]^2 \frac{d}{d\varepsilon} V(r, \varepsilon) dr. \quad (8)$$

Since, according to (1),  $(d/d\varepsilon)V(r, \varepsilon)$  is positive, this derivative is negative. Hence,  $\operatorname{tg} \delta(l, \varepsilon)$  is a monotonous decreasing function of  $\varepsilon$ , bounded below. Hence, it has a limit  $\operatorname{tg} \delta(l)$  for  $\varepsilon \rightarrow 0$ . So for  $l > kR$  all partial wave amplitudes are defined in the singular case and we have an explicit bound on their magnitude. For  $l < kR$  the situation is not so simple, but we shall not need to investigate this case.

Now, we shall repeat essentially the same argument for  $k^2 < 0$ . Here we do not need to put any limit on  $l$  because  $V_{\text{eff}}$  is always positive. We now normalize the solution as

$$u_{el}(r) = u_{0l}(r) + kf(l, \varepsilon)[v_{0l}(r) + iu_{0l}(r)] \quad (9)$$

for  $r \geq R$ , where

$$f(l, \varepsilon) = \frac{e^{i\delta(l, \varepsilon)} \sin \delta(l, \varepsilon)}{k}. \quad (10)$$

Notice that for  $k = i\kappa$ ,  $f(l, \varepsilon)$  is purely real and so  $u_{el}(R)$  is either purely real or purely imaginary (according to the parity of  $l$ ). Then  $u_{el}(r)$  is either purely real or purely imaginary and, again, cannot vanish in  $0 < r \leq R$ . Playing with this in exactly the same way as in the physical case, one gets

$$\begin{aligned} |f(l, \varepsilon)| &< \frac{1}{|k|} \int_0^R |u_{0l}(r)|^2 V_\varepsilon(r) dr \\ |f(l, \varepsilon)| &< \frac{1}{|k|} \int_0^R |u_{0l}(r)|^2 V(r) dr. \end{aligned} \quad (11)$$

Similarly, one can show again that for each  $l$   $(d/d\varepsilon)f(l, \varepsilon)$  has a constant sign and therefore the limit for  $\varepsilon \rightarrow 0$  of all partial waves exists.

## 2. BOUND ON THE FORWARD AMPLITUDE

In the physical region  $k^2 > 0$  we have from Eq. (7) an upper bound

on the partial wave amplitude for  $l > kR$ . For  $l < kR$  we shall content ourselves with the unitarity condition:

$$|e^{i\delta_l} \sin \delta_l| < 1. \quad (12)$$

Hence, we write the upper bound on  $|F_\varepsilon(k^2, \cos \theta = 1)|$  as:

$$|F_\varepsilon(k^2, \cos \theta = 1)| < \frac{1}{k} \sum_0^{kR} (2l+1) + \frac{1}{k^2} \sum_{kR}^\infty \int (2l+1) |u_{0l}(r)|^2 V(r) dr. \quad (13)$$

Let us make here the further assumption that  $V(r)$  is less singular than  $1/r^3$ . This is already an interesting case because the critical case is  $1/r^2$ . Then we can majorize the second series by a sum over all partial waves

$$\int \frac{V(r)}{k^2} \left[ \sum_0^\infty (2l+1) |u_{0l}(r)|^2 \right] dr. \quad (14)$$

The bracket, according to the standard expansion of a plane wave in partial waves [2] is nothing but:

$$\frac{1}{2} \int_{-1}^{+1} k^2 r^2 e^{ikr \cos \phi} e^{-ikr \cos \phi} d \cos \phi = k^2 r^2. \quad (15)$$

Or, alternatively, one could say that (14) is just the Born approximation for the full amplitude. Hence we get finally

$$|F_\varepsilon(k^2, \cos \theta = 1)| < \frac{(kR+1)^2}{k} + \int_0^R r^2 V(r) dr. \quad (16)$$

So we get on the forward scattering amplitude a bound which is independent of  $\varepsilon$  and grows like  $k$  for  $k \rightarrow \infty$ .

Similarly, one can sum the partial wave series for  $k^2 < 0$ . Here we do not have unitarity at our disposal, but fortunately inequality (11) holds for all partial waves. So we get

$$|F_\varepsilon(k^2 < 0, \cos \theta = 1)| < \frac{1}{|k^2|} \int V(r) \left[ \sum_0^\infty (2l+1) |u_{0l}(r)|^2 \right] dr.$$

Now, in analogy with (15) the bracket is just

$$\frac{1}{2} \int_{-1}^{+1} |k^2| r^2 e^{ikr \cos \phi} e^{-ik^* r \cos \phi} d \cos \phi$$



which is less than  $|k|^2 r^2 \exp 2|k|r$ .

So we get:

$$|F_\varepsilon(k^2 < 0, \cos \theta = 1)| < \exp(2|k|R) \int_0^R V(r) r^2 dr. \quad (17)$$

Again the bound is independent of  $\varepsilon$  and finite for finite negative  $k^2$ . Since on the other hand each partial wave amplitude has a limit for  $k^2 < 0$ ,  $\varepsilon \rightarrow 0$ . It follows that  $F_\varepsilon(k^2 < 0, \cos \theta = 1)$  has a limit for  $\varepsilon \rightarrow 0$ .

### 3. PROOF OF DISPERSION RELATIONS

For  $\varepsilon \neq 0$  we take for granted that dispersion relations can be written [3]:

$$F_\varepsilon(k^2, 1) - F_\varepsilon(k_0^2, 1) = \frac{k^2 - k_0^2}{\pi} \int_0^\infty \frac{\text{Im } F_\varepsilon(k'^2, 1) dk'^2}{(k'^2 - k^2)(k'^2 - k_0^2)} \quad (18)$$

where we choose  $k_0^2 < 0$ .

Now  $F_\varepsilon(k_0^2, 1)$  has a limit for  $\varepsilon \rightarrow 0$ , and

$$|F_\varepsilon(k^2, 1)| < |F_\varepsilon(k_0^2, 1)| + \frac{|k^2 - k_0^2|}{\pi} \int_0^\infty \frac{\frac{(k'R+1)^2}{k'} + \int_0^R r^2 V(r) dr}{|k'^2 - k^2| |k'^2 - k_0^2|} dk'^2 \quad (19)$$

so  $|F_\varepsilon(k^2, 1)|$  is bounded uniformly in  $\varepsilon$  in any finite region of the  $k^2$  plane which does not contain the cut  $k^2 = 0 \rightarrow k^2 = \infty$ ,  $F_\varepsilon(k^2, 1)$  has a limit for  $k^2 < 0$ ,  $\varepsilon \rightarrow 0$ . Hence, according to Vitali's theorem [4]  $F_\varepsilon(k^2, 1)$  has a limit for all  $k^2$  outside the cut and this limit  $F(k^2, 1)$  is analytic in the cut  $k^2$  plane. On the other hand, this limit, according to inequality (19), cannot grow faster than  $|k|^{1+\varepsilon}$  as  $|k|$  goes to infinity in complex directions. So there is no essential singularity at infinity in the physical sheet, and a dispersion relation holds for  $F(k^2, 1)$ , which is what we wanted to prove.

Improvements of this proof would be:

i) to accept higher singularities than  $1/r^3$ . This can be done for any power singularity without difficulty. For instance we should majorize

$$\sum_{kR}^\infty (2l+1) |u_{0l}|^2 \quad \text{for } kR > 1$$

by  $k^2 r^2 - (\sin kr)^2$  instead of  $k^2 r^2$  if we want to include singularities in  $1/r^{5-\varepsilon}$ .

ii) to include at intermediate distances an attractive region and possibly an exponentially decreasing tail. This can be done, by cutting the potential into various pieces but it requires a lot of epsilonics. It is clear that since in the non-singular case this makes no problem, it will not alter the result in the singular case.

#### ACKNOWLEDGEMENTS

I wish to thank Dr. K. Dietz who raised the question of the validity of Born approximation for singular potentials, which led me to the present work. I also wish to thank Professors R. Geballe, E. Henley and B. Jacobsohn who invited me to lecture on scattering theory at the University of Washington, where I had the idea to mix unitarity and Born approximation in potential scattering to get bounds on scattering amplitudes.

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# BOUNDARY CONDITIONS AND GENERAL RELATIVITY

O. KLEIN

*Stockholm*

*(Received April 24, 1965)*

The question of boundary conditions of Einstein's gravitational field equations has played a peculiar role in discussions about the content and true foundation of general relativity theory. Hence, the fact that the ordinary solutions of these equations, representing the gravitational field surrounding a limited distribution of matter, satisfy a condition at infinity ( $g_{ik} \rightarrow \text{constant}$ ), which is not covariant against arbitrary coordinate transformations, was regarded by Einstein as contradicting the spirit of his theory and was one of his main arguments behind his closed universe solution from which the branch of relativistic cosmology originated. On the other hand, the same fact has led Fock to deny the reality of general relativity, thereby assuming that a certain condition, limiting the choice of coordinates and fulfilled by the mentioned boundary condition, should be added as a necessary complement to Einstein's equations.

It is also a fact, however, that the equations expressing the physical laws in Einstein's theory *are* covariant against an arbitrary transformation of the four coordinates used for mapping the space-time region under consideration. And this fact is by no means trivial as is the introduction of curvilinear coordinates in pre-relativistic physics. The difference is that in relativity theory the  $g_{ik}$ , the coefficients of the Minkowskian quadratic form in general coordinates, are themselves field quantities like the gravitational potential of the Poisson equation. Still, *general relativity* is only half the foundation of the theory, the other half being the *equivalence principle*, which provides the physical interpretation of the mathematical formalism.

According to this principle the influence of gravitation on any physical phenomenon may be derived from the knowledge of the corresponding gravitationfree case of special relativity theory, this being

obtained by means of a locally, freely falling frame of reference in which the effects of gravitation are removed in the nearest neighbourhood of the space-time point under consideration. While this neighbourhood is, strictly speaking, infinitesimal, it may in practice be very large, namely as large as the gravitational field may be regarded as homogeneous and constant. An important consequence of the equivalence principle is that measurements – in principle – have to be carried out by tools at rest in the locally gravitationfree frame, the general coordinates having no physical meaning outside of their mapping role, a claim comparable with that to be fulfilled by the arrangements used in the observation of quantum phenomena.

As is well known, the Gaussian analytical geometry of curved surfaces and its generalization by Riemann to an arbitrary number of dimensions has been most helpful for the mathematical formulation of Einstein's ideas. From this viewpoint the removal of a gravitational field according to the equivalence principle appears as an analogy to the possibility of regarding an infinitesimal part of a curved surface as plane, i.e. describable by means of Euclidean geometry. It would seem that the exaggeration of this useful analogy is at the root of the controversial opinions just mentioned. Thus, while the introduction of curvilinear coordinates on a plane does not change any of the quantities which are of geometrical interest, the introduction of an accelerated frame of reference means the appearance of a gravitational field, which is locally indistinguishable from a "real" gravitational field, being, hence, just as physical as a magnetic field which may be removed by a Lorentz transformation. Although fields of non-vanishing curvature tensor cannot be entirely removed by means of a coordinate transformation and are thus in principle distinguishable from removable fields, a distinction between "real" and "unreal" fields by this criterion is certainly against the very essence of the equivalence principle.

Let us after these introductory remarks consider the boundary condition of an ordinary solution of Einstein's field equations such as that given by Schwarzschild and Droste for the field outside of a spherically symmetrical distribution of matter. This solution, which describes the motion of a particle (in practice a planet) around a fixed central body, when other influences may be neglected, is usually derived under the



assumption that the system – as far as the gravitational field due to itself may be neglected – is surrounded by an infinite empty space of Lorentz metric. From a mathematical point of view this way of simplifying the problem is practical and seemingly harmless. Still, it is probably responsible for the confusion of the problem of boundary conditions with the problem about the structure of the universe at large, to which the belief of Einstein and many of his followers in Mach's idea (that a body in an otherwise empty universe would have no inertia) may have contributed.

In order to show that there is no immediate relation between these two problems we shall consider the planetary system as a member of a large system, a galaxy, consisting of a great number of similar planetary systems, the central bodies of which are no longer fixed but free to move. Let us as a simplified model of such a galaxy assume the large system to be spherical and of constant average density, being, so to say, a gas, the molecules of which are planetary systems. The gravitational force at a distance  $r$  from the centre of the galaxy will then have the magnitude  $\frac{4}{3}\pi\gamma\mu r$ , where  $\mu$  is the average density and  $\gamma$  the gravitational constant. Then the lack of homogeneity of the galactic field relevant for the planetary system (being the fraction  $d/r$  of the field) will be  $\frac{4}{3}\pi\gamma\mu d$ , where  $d$  is the radius of the planetary system, independent of its position. On the other hand, the gravitational field due to the mass  $M$  of the planetary system at the distance  $d$  from its centre is equal to  $\gamma M/d^2$ . Hence, as far as the internal motions of the planetary system are concerned, the outward field will be practically homogeneous if  $\mu \ll 3M/4\pi d^3$ , a condition implying simply that the density of the galaxy is very small compared to the mean density of the planetary system.

With  $R$  being the radius and  $NM$  the mass of the galaxy, the condition in question may be written as  $N \ll (R/d)^3$ . Taking for  $N$  and  $R$  the approximate values for our galaxy ( $N \sim 10^{11}$ ,  $R \sim 10^5$  light years) and for  $d$  the distance from the sun to Pluto, the outermost planet ( $d \sim 10^{-3}$  light years), we see that the left side of the inequality is only about the fraction  $10^{-13}$  of the right side. More realistic assumptions about the structure of the galaxy would not, as is easily seen, change these orders of magnitude in any important way. Hence, the average galactic field is not only weak but of negligible inhomogeneity.

geneity as far as the solar system is concerned, the whole system falling freely in a practically homogeneous gravitational field. Consequently there will be no gravitational field in a frame of reference fixed to the centre of gravity of the solar system, as far as the gravitational field due to the masses of the system itself may be neglected. Hence, in this frame, which is just the Copernican one, the ordinary boundary condition (asymptotically constant  $g_{ik}$ ) is practically fulfilled, the meaning of the word "asymptotically" being now "at large distances from the system, which are yet small compared to the average distance between stars in the galaxy". Far from being contrary to the spirit of Einstein's theory, we see that the boundary condition in question has the same background as the equivalence principle.

We shall not here enter more closely on the bearing of these considerations on the cosmological problem. It should be mentioned, however, that a similar consideration may be carried out for the system of galaxies, whether it be the whole universe corresponding to one of the expanding cosmological solutions or simply one among a multitude of similar, limited systems. Thus, using the estimates of its expansion velocity and average density it may be shown, that for the relevant surroundings of a galaxy or galaxy cluster a frame of reference may be introduced, in which again the ordinary boundary conditions are valid "asymptotically" with sufficient approximation.

Finally it should be stressed that inertial forces appear in the usual way in a frame of reference which is accelerated with respect to an approximately gravitationfree frame of reference of the kind considered in the above examples, there being – contrary to Mach's idea – no immediate relation between the inertia of bodies or particles and the structure of the universe. It should also be remembered that energy and momentum of an approximately isolated system are defined with respect to a gravitationfree outward frame.

# PARITY AND MOMENTUM, A PRELUDE TO THE USE OF GROUP THEORY IN PHYSICS

HARRY J. LIPKIN

*The Weizmann Institute of Science*

*Rehovoth, Israel*

*(Received April 24, 1965)*

Consider a one-dimensional non-relativistic many-particle system whose interactions are invariant under translations and space inversion. The total momentum of the system  $K$  and the parity  $P$  are therefore conserved and commute with the Hamiltonian  $H$ .

$$[H, K] = 0 \quad (1a)$$

$$[H, P] = 0. \quad (1b)$$

However, momentum and parity do not commute. A momentum eigenstate does not have a definite parity, and a parity eigenstate does not have a definite momentum (except for the trivial case of zero momentum). Thus we can find solutions of the Schroedinger equation for this system which have *either a definite momentum, or a definite parity, but not both*. The solutions with a definite momentum are those in which the center of mass motion is described by a *traveling* plane wave;  $e^{iK'X}$ . The solutions with a definite parity are those in which the center of mass motion is described by a *standing* plane wave,  $\cos K'X$  or  $\sin K'X$ . The parity and momentum eigenstates having the same energy eigenvalue are clearly related by a simple linear transformation. The eigenvalue spectrum of the Hamiltonian is characterized by a twofold degeneracy. For each momentum eigenvalue  $K' > 0$ , there are two degenerate eigenfunctions of  $H$  having the form  $e^{iK'X}\varphi_\alpha$  and  $e^{-iK'X}\varphi_\alpha$  where  $\varphi_\alpha$  describes all the other degrees of the system except center of mass motion. We have chosen the momentum eigenstates. The corresponding parity eigenstates are  $\sin(K'X)\varphi_\alpha$  and  $\cos(K'X)\varphi_\alpha$ .

From this example we see that when a Hamiltonian commutes with

two operators which do not commute with one another, the eigenvalue spectrum of the Hamiltonian consists of *degenerate multiplets*. Furthermore, we can determine the characteristics of these multiplets *without knowing anything more about the details of the Hamiltonian and the dynamics of the system*. If the Hamiltonian is invariant under symmetry operations, such as space inversion or translations, then an eigenfunction of the Hamiltonian is transformed into another eigenfunction *with the same eigenvalue* by these transformations. If there are two non-commuting symmetry operations, like space inversion and translation, both operations cannot leave a state invariant (except for special cases like zero momentum), and therefore the successive operation with these transformations create new states which constitute a multiplet of degenerate eigenfunctions of the Hamiltonian. The properties of the multiplets which can arise are determined by the relations between the different non-commuting symmetry operations, and are independent of further properties of the Hamiltonian.

Let us now examine our particular example in more detail and show formally how many properties of the eigenfunctions of  $H$  follow from the interplay of the translation and space inversion transformations.

We first consider parity. The operator  $P$  satisfies the equation

$$P^2 = 1. \quad (2)$$

Its eigenvalues are thus  $\pm 1$ , called even and odd. Parity conservation helps in solving the Schroedinger equation, because we can look for simultaneous eigenfunctions of  $H$  and  $P$ . If we choose as a basis of functions for solving the Schroedinger equation a set which are already eigenfunctions of  $P$ , we cut our work in half, because the Hamiltonian cannot mix even and odd states. We have separated our Hilbert space into two pieces which are decoupled from one another.

We can also classify *operators* as even or odd under parity, according to whether they commute or anticommute with  $P$ . Any operator  $A$  can be written as the sum of an even part  $A_e$  and an odd part  $A_o$

$$A = A_e + A_o \quad (3a)$$

$$A_e = \frac{1}{2}(A + PAP) \quad (3b)$$

$$A_o = \frac{1}{2}(A - PAP). \quad (3c)$$



The even and odd operators satisfy the relations

$$PA_eP = A_e \quad (4a)$$

$$PA_oP = -A_o. \quad (4b)$$

Even and odd operators satisfy simple selections rules. Even operators have non-vanishing matrix elements only between states of the same parity; odd operators have non-vanishing matrix elements only between states of opposite parity. This is seen formally by considering the matrix elements between two states of parity  $P'$  and  $P''$

$$\langle P'|A_e|P''\rangle = \langle P'|PA_eP|P''\rangle = P'P''\langle P'|A_e|P''\rangle = 0 \quad \text{if } P' = -P'' \quad (5a)$$

$$\langle P'|A_o|P''\rangle = -\langle P'|PA_oP|P''\rangle = -P'P''\langle P'|A_o|P''\rangle = 0 \quad \text{if } P' = P''. \quad (5b)$$

There are simple rules for combining parities of different parts of a system. The parity of a complex system is just the product of the parities of its component parts.

We now consider momentum. The operator  $K$  does not satisfy any equation analogous to (2) and has a continuous spectrum of eigenvalues. Momentum conservation helps us in solving the Schroedinger equation, because we can look for simultaneous eigenfunctions of  $H$  and  $K$ . By choosing a basis of functions which are already eigenfunctions of  $K$ , we have reduced our work considerably, because the Hamiltonian cannot mix states having different eigenvalues of  $K$ . We have separated our Hilbert space into an *infinite number of pieces* which are decoupled from one another. In effect, we have removed one degree of freedom from the problem to be solved. Each decoupled subspace of the Hilbert space has one degree of freedom *less* than the original problem. For a one-particle problem, momentum conservation solves the Schroedinger equation completely, giving plane wave solutions.

Thus parity corresponds to a *finite* set of transformations (just space inversion), has a *finite* set of eigenvalues, and divides the Hilbert space up to a *finite* number of pieces. Momentum corresponds to a continuous group of transformations (translations), has a *contin-*

uous spectrum of eigenvalues, and separates a degree of freedom from the problem.

In the same way that operators can be divided into two types, corresponding to their behaviour under space inversion, they can be divided into a continuous infinity of types, corresponding to their behaviour under translations. The expansion of an arbitrary operator  $A$  into a continuous set of operators  $A_{K'}$ , analogous to the parity expansion (3) is just a Fourier expansion:

$$A = \int dK' A_{K'} \quad (6a)$$

$$A_{K'} = \frac{1}{2\pi} \int dx [e^{i(K-K')x}, A]. \quad (6b)$$

The operators  $A_{K'}$  satisfy the relation analogous to the eigenvalue equation

$$[K, A_{K'}] = K' A_{K'}. \quad (7)$$

The operators  $A_{K'}$  have the property of adding a momentum  $K'$  to a state. Simple examples of such operators are  $e^{iK'x_j}$ ,  $p_k e^{iK'x_j}$  and  $(e^{iqx_i}) \cdot (e^{i(K'-q)x_j})$  where  $x_i$ ,  $x_j$ ,  $p_i$  and  $p_j$  are co-ordinates and momenta of two particles in the many-particle system and  $q$  is arbitrary. These "momentum eigenoperators" satisfy momentum conservation selection rules, analogous to the parity selection rules satisfied by the "parity eigenoperators" [5]. The matrix elements of the operators  $A_{K'}$  between two momentum eigenstates  $|K''\rangle$  and  $|K'''\rangle$  vanish unless momentum is conserved. This can also be seen from the formal properties (5) of the operators:

$$\begin{aligned} K' \langle K''' | A_{K'} | K'' \rangle &= \langle K''' | [K, A_{K'}] | K'' \rangle = (K''' - K'') \langle K''' | A_{K'} | K'' \rangle \\ &= 0 \quad \text{unless } K' = K''' - K''. \end{aligned} \quad (8)$$

There are also rules for combining momenta of different parts of a system. The momentum of a complex system is just the sum of the momenta of the component parts.

The fun begins when we consider parity and momentum together, because  $P$  and  $K$  do not commute. They anticommute

$$PK = -KP. \quad (9)$$

This immediately leads to the doublet structure of the eigenfunctions of  $H$ . Let  $\psi_{K'}$  be an eigenfunction of  $H$  and  $K$  with eigenvalues  $E$  and  $K'$

$$H\psi_{K'} = E\psi_{K'} \quad (10a)$$

$$K\psi_{K'} = K'\psi_{K'}. \quad (10b)$$

Then we can define a corresponding state

$$\psi_{-K'} = P\psi_{K'}. \quad (11)$$

This state (11) is degenerate with the state (10) and has the opposite momentum eigenvalue

$$H\psi_{-K'} = HP\psi_{K'} = PH\psi_{K'} = EP\psi_{K'} = E\psi_{-K'} \quad (12a)$$

$$K\psi_{-K'} = KP\psi_{K'} = -PK\psi_{K'} = -K'P\psi_{K'} = -K'\psi_{K'}. \quad (12b)$$

Since  $P^2 = 1$ , further operation with  $P$  brings us back where we came from, and we obtain no new states. We thus see that the eigenfunctions of  $H$  separate into degenerate multiplets each characterized by a number  $K' \geq 0$ , and that the multiplets are doublets if  $K' \neq 0$  and singlets if  $K' = 0$ . In the representation we have chosen, the operator  $K$  is diagonal, while the operator  $P$  is “almost diagonal”; i.e.  $P$  has non-vanishing matrix elements *only between states within the same multiplet*. We could have chosen a representation in which  $P$  would be diagonal; then  $K$  would be “almost diagonal”.

Let us specify the eigenfunctions of  $H$  by the following quantum numbers: the magnitude of the momentum  $K'$ , the *sign* of the momentum,  $\sigma' = \pm 1$ , and a set of quantum numbers  $\alpha'$  which specify the other degrees of freedom of the system. The quantum number  $K'$  specifies the *kind* of multiplet containing the state. The quantum number  $\sigma'$  specifies the particular state within the multiplet. The matrix elements of the operators  $K$  and  $P$  are completely specified in this representation

$$K|K', \sigma', \alpha'\rangle = \sigma' K' |K', \sigma', \alpha'\rangle \quad (13a)$$

$$P|K', \sigma', \alpha'\rangle = |K', -\sigma', \alpha'\rangle. \quad (13b)$$

The operator  $K$  is diagonal with the eigenvalue  $\sigma' K'$ . The operator

$P$  is "almost diagonal", with matrix elements of magnitude unity between states of the same multiplet.

Instead of choosing the sign of the momentum to specify the state within the multiplet, we can choose the parity. For this case,

$$K|K', P', \alpha'\rangle = K'|K', -P', \alpha'\rangle \quad (14a)$$

$$P|K', P', \alpha'\rangle = P'|K', P', \alpha'\rangle. \quad (14b)$$

Here  $P$  is diagonal and  $K$  is almost diagonal.

The parity and momentum "eigenoperators" also form multiplets, when we consider parity and momentum together. For each operator  $A_{K'}$  satisfying equation (7), we can define a companion  $A_{-K'}$ .

$$A_{-K'} = PA_{K'}P \quad (15a)$$

$$[K, A_{-K'}] = [K, PA_{K'}P] = -K'A_{-K'}. \quad (15b)$$

The operator multiplets  $(A_{K'}, A_{-K'})$  have a structure resembling the wave function multiplets.

We shall now find a very important relation between matrix elements of operator multiplets between sets of state belonging to multiplets. Consider the matrix element

$$\langle K', \sigma', \alpha' | A_{K'''\sigma'''} | K'', \sigma'', \alpha'' \rangle. \quad (16)$$

If we consider all the matrix elements of the two components of the operator multiplet  $A_{K'''\sigma'''}$  between states of the multiplets  $(K', \alpha')$  and  $(K'', \alpha'')$  there are a total of eight independent matrix elements. We shall see that these are all proportional to a *single quantity* depending upon the properties of the system, with proportionality factors depending only upon the algebra of the operators  $P$  and  $K$ . First we note that no more than two of the eight matrix elements can differ from zero, because the momentum conservation relation (8) requires that  $K'\sigma' = K''\sigma'' + K'''\sigma'''$ . The two non-vanishing matrix elements are equal, since,

$$\begin{aligned} \langle K', \sigma', \alpha' | A_{K'''\sigma'''} | K'', \sigma'', \alpha'' \rangle &= \\ &= \langle K', \sigma', \alpha' | P(PA_{K'''\sigma'''}P)P | K'', \sigma'', \alpha'' \rangle \\ &= \langle K', -\sigma', \alpha' | A_{K'''\sigma'''} | K'', -\sigma'', \alpha'' \rangle. \end{aligned} \quad (17)$$



We can thus write

$$\begin{aligned} \langle K'\sigma'\alpha' | A_{K'''\sigma'''} | K''\sigma''\alpha'' \rangle &= \\ &= V(K', \sigma', K'', \sigma'', K'''\sigma''') \langle K'\alpha' | A_{K'''} | K''\alpha'' \rangle \end{aligned} \quad (18)$$

where  $V(K', \sigma', K'', \sigma'', K'''\sigma''')$  is a coefficient depending only upon the parity and momentum quantum numbers  $(K', K'', K''', \sigma', \sigma'', \sigma''')$  and independent of the other quantum numbers  $(\alpha', \alpha'')$  and the particular nature of the operator  $A$ . The double-barred “reduced matrix element”  $\langle K'\alpha' | A_{K'''} | K''\alpha'' \rangle$  depends only upon the multiplets, but is independent of the quantum numbers which specify the particular members of the multiplets. In this case all the coefficients  $V$  vanish except for the two corresponding to values of the argument which satisfy momentum conservation, and  $V = 1$  for these cases. This result (18) is a simplified version of Wigner-Eckart theorem.

In this simple example we have seen how certain symmetry properties of a Hamiltonian lead to many useful results. This can be summarized in a form which has a more general validity:

Whenever the Hamiltonian of a physical system is invariant under two or more transformations which do not commute with one another, one can define a group of non-commuting transformations (a set of non-commuting operators) under which the Hamiltonian is invariant. In this case:

1. The eigenvalue spectrum of the Hamiltonian consists of degenerate multiplets.
2. The structure of the possible multiplets (singlet and doublets in the parity-momentum example,  $2j+1$ -plets for angular momentum) is determined completely by the relations of the transformations among themselves and is independent of the detailed properties of the Hamiltonian.
3. The Hamiltonian can be diagonalized in a representation in which all the operators of the group are either diagonal or “almost diagonal”, they have non-zero matrix elements only between states which are members of the same degenerate multiplet. The matrix elements of these operators are determined completely by the algebra of the operators and are independent of the specific details of the Hamiltonian.

4. "Operator multiplets", generally called "irreducible tensor operators" can be defined by analogy with the wave function multiplets. These have the same structure as the wave function multiplets.

5. The matrix elements of members of a given irreducible tensor operator between states of two multiplets are all proportional to one (in some special cases more than one) reduced matrix element which is independent of the quantum numbers specifying the particular member of the multiplet. The coefficient is independent of the details of the wave functions and operators and depends only on the quantum numbers associated with the symmetry group. These are called Wigner coefficients or generalized Clebsch-Gordan coefficients.

6. There are simple rules for combining multiplets which depend only upon the algebra of the group.

# POLARIZATION AND ZEROS OF THE SCATTERING AMPLITUDE \*

A. DE-SHALIT

*Department of Nuclear Physics, The Weizmann Institute of Science  
Rehovoth, Israel*

*(Received April 24, 1965)*

Diffraction phenomena have been observed in many scattering and reaction processes both in the realm of nuclear physics and in that of elementary particle physics. An extensive analysis of such processes has been carried out. In some cases detailed theories have been used, and in others one was satisfied with more crude approximations, such as the Blair-Drozdov strong absorption model. The purpose of the present remarks is to look at diffraction-like processes from yet another point of view, which, it is believed, may be helpful in clarifying some regularities observed in the polarization accompanying some scattering and reaction processes.

For the sake of definiteness let us confine ourselves to elastic-scattering, and consider first the scattering of a zero-spin projectile on a zero-spin target. For a given center-of-mass energy, the scattering amplitude is then a single complex function  $f(z)$  of the center-of-mass scattering angle  $z = \cos \theta$ , and the differential cross-section is given by

$$\frac{d\sigma}{d\Omega} = |f(z)|^2. \quad (1)$$

The function  $f(z)$  is physically meaningful only for real values of  $z$  satisfying  $-1 < z < 1$ . It is, however, convenient to continue it analytically into the complex  $z$ -plane and study its properties in the neighbourhood of the real axis. For diffraction-like scattering, especially when the "diffraction minima" are deep, one is led naturally to the study of the zeros of  $f(z)$  [1]. Indeed, if one uses a parabolic approximation for  $d\sigma/d\Omega$  in the vicinity of a minimum of the differ-

\* The research reported in this document has been sponsored in part by the National Bureau of Standards.

ential cross-section one obtains

$$\frac{d\sigma}{d\Omega} \approx \text{const. } |z - z_i|^2 \quad (2)$$

where  $z_{i0} = \text{Re } z_i$  is the position of the minimum considered and  $z_i$  (or  $z_i^*$ ) is a zero of the scattering amplitude. The imaginary value of  $z_i$  can be obtained from the approximate expression (2) for the differential cross-section by evaluating the actual value of  $d\sigma/d\Omega$  at  $z_{i0}$  and its curvature there. For sharp deep minima typical values of  $\text{Im } z_i$  can be as small as 0.1 or even 0.01.

The number of zeros of  $f(z)$  which are close to the real axis is related to the highest  $l$ -value of the partial wave contributing significantly to the scattering process. To see this, let us expand  $f(z)$  in a series of Legendre polynomials

$$f(z) = \frac{i}{2k} \sum_0^{\infty} (2l+1)(1-\eta_l)P_l(z). \quad (3)$$

Here  $k$  is the wave-number of the scattered particle and the complex numbers  $\eta_l$  are related to the complex phase-shifts  $\delta_l$  through

$$\eta_l = e^{2i\delta_l}. \quad (4)$$

$1 - \eta_l = 0$  signifies no contribution from that particular  $l$ -channel to the scattering. If now there exists an  $L$  such that for every  $l > L$

$$(2l+1)|1 - \eta_l| \ll 1$$

but

$$(2L+1)|1 - \eta_L| \not\ll 1 \quad (5)$$

then there are at most  $L$ -zeros of  $f(z)$  close to the real axis. Indeed  $f_L(z)$  defined by

$$f_L(z) = \frac{i}{2k} \sum_0^L (2l+1)(1-\eta_l)P_l(z) \quad (6)$$

has got exactly  $L$  zeros, being a polynomial of degree  $L$  in  $z$ . Consider now

$$f_{L+1}(z) = \frac{i}{2k} \sum_0^{L+1} (2l+1)(1-\eta_l)P_l(z)$$



$$\begin{aligned}
&= f_L(z) + \frac{i}{2k} [2(L+1)+1](1-\eta_{L+1})P_{L+1}(z) \\
&= \frac{i}{2k} [a_{L+1}z^{L+1} + a_Lz^L + \dots]. \quad (7)
\end{aligned}$$

$f_{L+1}(z)$  has got  $(L+1)$  zeros  $z_i^{(L+1)}$ ;  $L$  of these, as is seen from (7), will be close to the zeros of  $f_L(z)$ , especially if  $|1-\eta_{L+1}|$  is small enough. The position of the "extra" zero of  $f_{L+1}(z)$  can be estimated by noting that

$$\sum z_i^{(L+1)} = -\frac{a_L}{a_{L+1}}. \quad (8)$$

From (5) it follows that  $|a_L/a_{L+1}| \gg 1$ , so that for small values of  $|1-\eta_{L+1}|$  the "extra" zero lies far from the real axis and would not reflect itself in an extra minimum in  $d\sigma/d\Omega$  along the real axis.

There are therefore *at most*  $L$  zeros of  $f(z)$  which lie close to the real axis, where  $L$  is determined by (5). The neighbourhood of the real axis which contains all the zeros of  $f_L(z)$  defined by (6) can be determined using Rouché's theorem. In fact,  $P_l(z)$  satisfies

$$(2l+1)zP_l(z) = (l+1)P_{l+1}(z) + lP_{l-1}(z)$$

and hence

$$\begin{aligned}
|z| \sum_0^{L-1} (2l+1)|P_l(z)| &\leq \sum_0^{L-1} (l+1)|P_{l+1}(z)| + \sum_0^{L-1} l|P_{l-1}(z)| \\
&= \sum_0^L l|P_l(z)| + \sum_0^{L-2} (l+1)|P_l(z)| \\
&= \sum_0^{L-1} (2l+1)|P_l(z)| + L\{|P_L(z)| - |P_{L-1}(z)|\}. \quad (9)
\end{aligned}$$

Hence

$$(|z|-1) \sum_0^{L-1} (2l+1)|P_l(z)| \leq L|P_L(z)|. \quad (10)$$

Since  $|\eta_l| < 1$  we now obtain, for  $|z|-1 > 0$ ,

$$\left| \sum_0^{L-1} (2l+1)(1-\eta_l)P_l(z) \right| \leq 2 \sum_0^{L-1} (2l+1)|P_l(z)| \leq \frac{2L}{|z|-1} |P_L(z)|.$$

We now chose  $|z|$  big enough so that

$$\frac{2L}{|z|-1} \leq (2L+1)|1-\eta_L|$$

i.e.

$$|z| \geq \frac{2L}{2L+1} \frac{1}{|1-\eta_L|} + 1. \quad (11)$$

For such values of  $|z|$  we obtain:

$$|(2L+1)(1-\eta_L)P_L(z)| \geq \sum_0^{L-1} (2l+1)(1-\eta_l)P_l(z). \quad (12)$$

It follows then from Rouché's theorem that all the zeros of  $f_L(z)$  are contained within the circle defined by  $|z| = 1 + \{2L/(2L+1)(|1-\eta_L|)\}$ .

It is remarkable that the zeros of  $f_L(z)$  can be localized within a circle of such relatively small radius (provided  $|1-\eta_L| \ll 1$ ; see Eq. (5)). As we see, the only important elements used in the derivation of this result are the unitarity requirement  $|\eta_l| \leq 1$ , and the cut-off in  $l$ -space given by (5). We also see that if we were to carry out the same analysis for  $f_{L+1}(z)$ , the radius of the circle which includes all the  $L+1$  zeros of  $f_{L+1}(z)$  will be very large on account of the smallness of  $|1-\eta_{L+1}|$ . This is in agreement with our conclusion following Eq. (8).

Having thus studied some of the properties of the zeros of the scattering amplitude, let us now pass to the case of spin- $\frac{1}{2}$  particles elastically scattered by a spin-0 target. In this case the scattering amplitude is given by

$$M(z) = f(z)\mathbf{1} + g(z)\boldsymbol{\sigma} \cdot \mathbf{n}_s. \quad (13)$$

Here  $\mathbf{n}_s$  is the normal to the scattering plane,  $\boldsymbol{\sigma}$  is the spin matrix operating on the projectile's spinor, and  $\mathbf{1}$  is the unit matrix;  $f(z)$  and  $g(z)$  are two complex functions. The differential cross-section for an unpolarized beam is given by

$$\frac{d\sigma}{d\Omega} = |f(z)|^2 + |g(z)|^2, \quad (14)$$

and the polarization of an initially unpolarized beam undergoing a

scattering by an angle  $\theta = \arccos z$  is given by

$$P(z) = \frac{2 \operatorname{Re} f^*(z)g(z)}{d\sigma/d\Omega}. \quad (15)$$

For most angles  $\theta$  it is true that  $|f(z)| \gg |g(z)|$ , so that the gross structure of the differential cross-section is determined by  $|f(z)|^2$ , minima of  $d\sigma/d\Omega$  occurring, again, at real values of  $z$  closest to the (complex) zeros of  $f(z)$ .

Suppose  $z_i$  is such a zero of  $f(z)$ , lying close to the real axis and thus giving rise to a sharp, deep minimum in  $d\sigma/d\Omega$ , and let us investigate the behaviour of  $P(z)$  near  $z_i$ . Ignoring the variation of  $g(z)$  in the immediate vicinity of  $z_{i0} = \operatorname{Re} z_i$ , and putting in this neighbourhood

$$g(z) \approx G, \quad f(z) \approx (z - z_i)F,$$

we have

$$P(z) \approx 2 \frac{\operatorname{Re} [(z - z_i)^* F^* G]}{|(z - z_i)F|^2 + |G|^2}. \quad (16)$$

The whole variation of the polarization  $P$  with  $\theta$  around  $\theta_i = \arccos z_{i0}$  is contained in  $z - z_i$ . If  $F^*G$  is pure real, then

$$P(z) = \frac{2(F^*G) \operatorname{Re} (z - z_i)}{|z - z_i|^2 |F|^2 + |G|^2}$$

and we see that the polarization goes through zero at  $z = z_{i0}$  and that its slope at  $z = z_{i0}$  increases as  $z_i$  comes close to the real axis. Actually

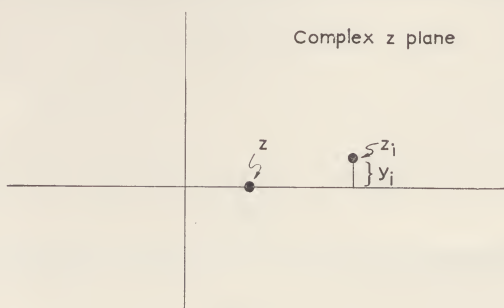


Fig. 1.

if we put  $z = x + iy$  and  $z_i = x_i + iy_i$ , we can see that for a variation  $\Delta x \approx \pm 3y_i$  around  $x_i$ , the polarization changes essentially from its

maximal value in one direction to its maximal value in the opposite direction. If  $F^*G$  were pure imaginary, the polarization would have not changed sign around  $z_{i0} = \text{Re } z_i$ ; as a matter of fact  $P(z)$  would have had a maximum at  $z_{i0}$ .



Fig. 2. Differential cross-section for 64.3 MeV  $\alpha$ -particles scattered on  $\text{Fe}^{58}$  (Ref. 3).

In the general case, when  $F^*G$  is neither real nor pure imaginary, the polarization will change sign around  $z_{i0}$ , the point at which  $P(z)$  vanishes being closer to  $z_{i0}$  the closer  $F^*G$  is to being pure real.



It has been noted already some time ago [2] that the polarization in elastic scattering looks like a logarithmic derivative of the angular distribution. In particular this implies that the polarization vanishes at the minima of  $d\sigma/d\Omega$ , or, in our language, that  $F^*G$  is essentially real. We shall not go here into the properties of  $F^*G$ , but rather *assume* it to be real for relatively low energies. This, then, enables us to analyze differential cross-sections and the polarization corresponding to them

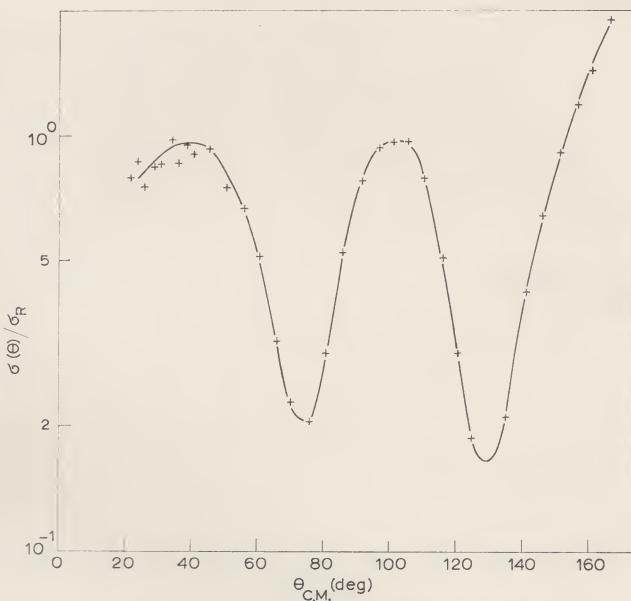


Fig. 3. Differential cross-section for 14.1 MeV proton scattered on Fe (Ref. 4).

in terms of the positions of the zeros of the scattering amplitude  $f(z)$  in the vicinity of the real axis.

To illustrate the relation between the number of minima in the differential cross-section and the largest significant  $l$ -value in the scattering we reproduce the data [3] of elastic scattering of  $\alpha$ -particles of 64.3 MeV on  $\text{Fe}^{58}$ . Estimating  $L_{\max}$ , the highest significant  $l$ -value, through

$$L_{\max} = kr_0 A^{\frac{1}{3}}$$

where  $k$  is the wave-number of the relative motion, we obtain for this

case  $L_{\max} \approx 20$ ; fig. 2 shows clearly 9 minima between  $10^\circ$  and  $85^\circ$ , which is about as many as could have been expected in this region if the maximum number of minima close to the real axis is not to exceed 20.

Fig. 3 shows the differential cross-section for the elastic scattering [4] of 14.1 MeV protons on Fe, while Fig. 4 shows the polarization of 14.5 MeV protons [5] scattered from  $\text{Fe}^{58}$ . It is seen that, around

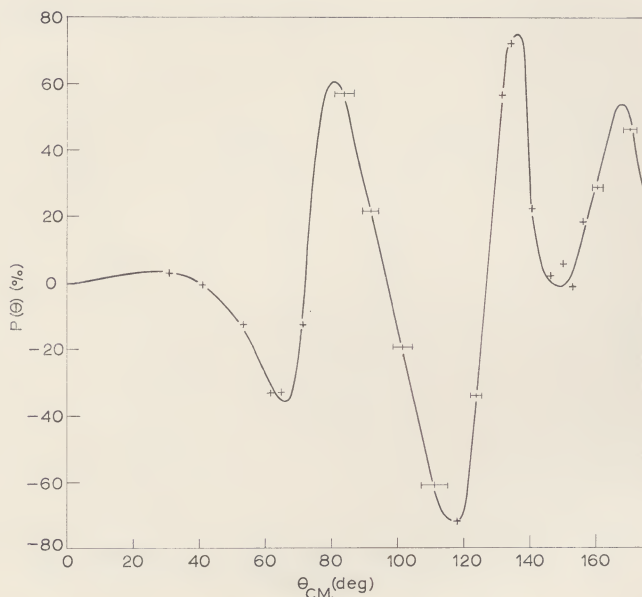


Fig. 4. Polarization of 14.5 MeV protons scattered on  $\text{Fe}^{58}$  (Ref. 5).

the minima of  $d\sigma/d\Omega$ , the polarization changes very rapidly, going for example, from  $-70\%$  at  $120^\circ$  to  $+70\%$  at  $134^\circ$ . From the shape of  $d\sigma/d\Omega$  around  $130^\circ$ , it is possible to conclude that the corresponding zero of the scattering amplitude lies at most a distance of 0.07 from the real axis (in the complex  $z = \cos \theta$  plane); this then suggests that the polarization will change from its minimum value to its maximum value over a range of about 0.2 in  $\cos \theta$  centered around the minimum of  $d\sigma/d\Omega$ . An inspection of Fig. 4 shows this actually to be the case.

It is impossible to present within this short note a complete analysis

of available data. I would like only to stress that the rapid variation of  $P(\theta)$  with  $\theta$  around the minima of  $d\sigma/d\Omega$ , previously interpreted [6] in terms of a diffraction mechanism, seems now to be more directly connected with the mere presence of deep minima in  $d\sigma/d\Omega$  without particular reference to the mechanism which produces these minima.

The remarks presented above are far from being complete, nor is it clear that they can be developed beyond offering qualitative explanations of some features of the differential cross-section and their relation to the associated polarization. It was nevertheless considered worthwhile to mention them since the zeros of analytic functions have been very extensively studied by the mathematicians, and it is not impossible that some of their results may turn out to be of relevance to the scattering problem.

I am indebted to Mr. A. Gersten for many stimulating discussions.

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# STRONGLY INTERACTING PARTICLES AND THE TRIPLET HYPOTHESIS

L. VAN HOVE

*CERN, Geneva*

*(Received April 25, 1965)*

Unless nature deceives us shrewdly, the physics of strong interactions is entering a new phase. The structure of baryons and mesons becomes analyzable in terms of concrete composite models which combine intuitive simplicity and successful power of prediction. Such a development, reminiscent of the discovery of powerful model theories for nuclear reactions in 1936 and nuclear structure in 1950, must mean much to Viki Weisskopf, whose life most brilliantly and productively combines a career of nuclear physics research with the scientific leadership of a top-rank laboratory for elementary particle physics.

What does this development prelude to? We cannot say yet, but we know enough to imagine a few concrete possibilities. It is fascinating to speculate on the resulting picture of hadron physics. Although present guesses are not likely to be the right ones, they may lead to useful questions and view-points.

## BARYONS AND MESONS AS COMPOSITE PARTICLES

The  $SU(3)$  and  $SU(6)$  symmetry theories have revealed many remarkable regularities among mesons and baryons. To visualize them intuitively one must imagine that the mesons are bound states  $m = t\bar{t}'$  of spin  $\frac{1}{2}$  particles  $t$  and  $\bar{t}'$ ,  $t$  and the antiparticle  $t'$  of  $\bar{t}'$  belonging to representation 3 of  $SU(3)$ , and similarly that the baryons are bound states  $b = tt't''$  of three spin  $\frac{1}{2}$  particles  $t$ ,  $t'$  and  $t''$  also belonging to representation 3 of  $SU(3)$ . If one accepts that the basic particles  $t$ ,  $t'$ ,  $\dots$  have fractional electric charge, one can assume them to belong to one single  $SU(3)$  triplet (by triplet we mean representation 3 of  $SU(3)$ ); they are then the "quarks" introduced by Gell-Mann and Zweig [1]. If one believes that all particles, even the basic ones, have electric charge 0 or  $\pm 1$ , the basic particles form at least two triplets;



the simplest model has then exactly two spin  $\frac{1}{2}$  triplets, one  $(T^+, T^0, T'^0)$  with baryon number  $N = 1$  and one  $(\bar{\Theta}^0, \bar{\Theta}^-, \bar{\Theta}'^-)$  with  $N = -1$ ; we call the  $T$  and  $\Theta$  "trions" [2].

The basic triplets, quarks or trions, are believed to be very heavy (several proton masses), so that mesons and baryons are tightly bound states with a binding energy almost as large as the total mass of the constituent triplets. This is usually advanced as a possible explanation of the success of mass splitting formulae in first order of perturbation. It may also explain in simple, intuitive terms a more familiar feature of hadron physics (hadron = strongly interaction particle), namely the existence of very many resonant states which, although they can decay through strong interactions, have a width small compared to the energy available for strong decay.

We are so used to this fact that we now regard it as perfectly normal. We also are no longer surprised that, even at relatively high energies, hadron collisions mostly produce a small number of resonances. Still, it should be said that these properties of hadrons are definitely unexpected in a theoretical framework where baryons and mesons are elementary particles. Bootstrap theory, even if it becomes one day quantitatively successful, may find it difficult to explain them convincingly because it postulates them from the very start. In contrast, they become very natural in the composite model of baryons and mesons. Resonances are then excited states of motion of the constituent triplets inside the baryon or meson. These states would be stable for strong interactions were it not that virtual triplet-antitriple pairs can be created and can energetically lead to decay when they are in the tightly bound configurations forming the low mass mesons. Such configurations are obviously exceptional, hence the relatively small width ( $\Gamma \lesssim 100$  MeV) of resonances [3]. Also the dominance of resonance production in strong interaction collisions becomes plausible even at relatively high energies.

The problem of nuclear forces comes to stand in another light if the baryons have a composite structure. They become analogous to interatomic forces: a strong repulsion at short distance, due to the exclusion principle for quarks or trions, and a van der Waals type force over a longer range  $m_\pi^{-1}$  characterized by the lightest meson [4]. If nature is made in this way, we need no longer be so surprised that

the nuclear forces appear to be too complicated to result directly from a simple basic theory. They might be indirect manifestations of the strong interaction between triplets in the same way as intermolecular forces are complicated consequences of the basically simple electromagnetic interaction between electrons and nuclei.

#### ELECTROMAGNETIC STRUCTURE OF HADRONS

Another consequence of the composite structure of mesons and baryons is that the electric charge and magnetic moment distributions of these particles can have no point singularity, or in other words that the electromagnetic form factors tend rapidly to zero for large momentum transfer. What can be said about the electromagnetic structure of the fundamental triplets? The remarkable feature is that they need not be stable for strong interactions, so that their point structure can be smeared out by the fact that they are no more than strong interaction resonances between composite particles. The latter property is indeed the simplest to assume for trions, all of which can decay strongly in the known baryons and mesons [5]. The situation is more complicated in the case of quarks where conservation of electric charge requires that at least one particle of fractional charge  $\frac{1}{3}$  or  $\frac{2}{3}$ , and its antiparticle, be absolutely stable. These stable particles need not be quark states, however. They can be bound states of  $n$  quarks and  $n'$  antiquarks where  $n - n'$  is not congruent to zero modulus three.

As to the form of the electromagnetic interaction, one would naturally expect it to be of the minimal type for the triplets, i.e., to be given by the following interaction term in the Lagrangian

$$L_{\text{em}} = e_0 J_{\mu}^{\text{em}} A_{\mu} \quad (1)$$

$$J_{\mu}^{\text{em}} = \sum_a Q_a \bar{\psi}_a \gamma_{\mu} \psi_a \quad (2)$$

where  $e_0$  is the proton charge,  $A_{\mu}$  the electromagnetic four-potential,  $\psi_a$  the Dirac field operator for basic triplet member  $a$  and  $Q_a$  its charge in units of  $e_0$ . All  $e_0 Q_a$  are of course real (hermiticity of (1) requires this), and the electromagnetic interaction is invariant for  $C, P, T$ , and conserves  $Q$  (total charge) and  $I_3$  (3d isospin component).

## WEAK INTERACTIONS OF HADRONS

We discuss the weak interaction between hadrons and leptons by adopting for it the familiar current  $\times$  current form

$$L_{\text{whl}} = \frac{G}{\sqrt{2}} (j_\mu^\dagger J_\mu + J_\mu^\dagger j_\mu) \quad (3)$$

$$j_\mu = \bar{\psi}_{(\nu e)} \Gamma_\mu \psi_{(e-)} + \bar{\psi}_{(\nu \mu)} \Gamma_\mu \psi_{(\mu-)} \quad (4)$$

$$\Gamma_\mu = \gamma_\mu (1 + \gamma_5). \quad (5)$$

$j_\mu$  is the leptonic current, and  $G$  is the Fermi constant appearing in leptonic weak interactions

$$L_{\text{wll}} = \frac{G}{\sqrt{2}} j_\mu^\dagger j_\mu.$$

For the hadronic current  $J_\mu$ , which carries  $\Delta Q = 1$ , we try simple expressions in terms of the Dirac field operators of the basic triplets.

This is particularly easy in the case of quarks. Denote the quark fields by  $\psi_1, \psi_2, \psi_3$ , the corresponding charges and isospins being  $\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}$  and  $\frac{1}{2}, \frac{1}{2}, 0$  respectively. The natural ansatz for the hadronic current is

$$J_\mu = c_0 \bar{\psi}_1 \Gamma_\mu \psi_2 + c_1 \bar{\psi}_1 \Gamma_\mu \psi_3 \quad (6)$$

The first term represents ( $\Delta I = 1, \Delta Y = 0$ ) transitions, the second one ( $\Delta I = \frac{1}{2}, \Delta Y = 1$ ). Universality in the sense of Cabibbo means

$$|c_0|^2 + |c_1|^2 = 1. \quad (7)$$

Hermiticity of (3) requires only  $G$  to be real,  $c_0$  and  $c_1$  can be arbitrary complex numbers. They can be made real, however, by applying appropriate gauge transformations

$$\exp(i\theta Q), \quad \exp(i\theta' I_3) \quad (8)$$

which leave strong and electromagnetic interactions invariant. As a consequence the hadron-lepton interaction  $L_{\text{whl}}$  obtained with (6) is invariant for  $CP$  and  $T$ . The same will hold for non-leptonic weak interactions of hadrons if they are generated by  $J_\mu^\dagger J_\mu$  or special terms of this expression (mainly the  $SU(3)$  octet part).

Let us try to generalize the above considerations to the trion case.

The following procedure leads to simple results. We construct the  $\Delta Y = 0$  part of  $J_\mu$  by taking it to belong to the ( $\Delta Q = 1$ ,  $\Delta I = 1$ ) generator of the isospin group (which is  $I_1 + iI_2$ ). Its expression is found to be

$$\sqrt{2}J_\mu^{(0)} = c_0(\overline{T^+}\Gamma_\mu T^0 - \overline{\Theta^+}\Gamma_\mu \Theta^0) \quad (9)$$

where the symbol of a particle represents its Dirac field operator. For the  $\Delta Y = 1$  part  $J_\mu^{(1)}$ , we take the  $\Delta Q = 1$ ,  $\Delta I = \frac{1}{2}$  generators of a larger internal symmetry group which can be introduced to describe the properties of the six trions [2]. Considering the two rank three Lie groups which have a basic representation of dimension six,  $SO(6)$  and  $Sp(6)$ , we find two such generators in each case, and  $J_\mu^{(1)}$  is taken as a superposition of the two corresponding terms

$$\sqrt{2}J^{(1)} = c_1(\overline{T^+}\Gamma_\mu T'^0 - \overline{\Theta'^+}\Gamma_\mu \Theta^0) + c'_1(\overline{\Theta^+}\Gamma_\mu T'^0 \pm \overline{\Theta'^+}\Gamma_\mu T^0). \quad (10)$$

The  $+$  ( $-$ ) in the last term corresponds to  $Sp(6)$  ( $SO(6)$ ). In the case of  $SO(6)$  the three independent terms in (9) and (10) correspond to the three generators of the group having  $\Delta Q = 1$ . For  $Sp(6)$  there are additional  $\Delta Q = 1$  generators having  $\Delta I = 0$  or  $1$ ; they seem to be unsuited for weak interactions. Universality would now probably be expressed through

$$|c_0|^2 + |c_1|^2 + |c'_1|^2 = 1. \quad (11)$$

The success of the Cabibbo analysis for leptonic decays of hadrons suggests that  $|c'_1|$  must be appreciably smaller than  $|c_1|$ .

By means of gauge transformations (8) one can make the coefficients  $c_0$ ,  $c_1$  real.  $c'_1$  can also be made real if there is, commuting with (8), a third gauge group for which strong and electromagnetic interactions are invariant. The natural candidate is

$$\exp(i\theta''Y) \quad (12)$$

$Y$  being the  $SU(3)$  hypercharge. For trions it is connected to  $Q$  and  $I_3$  by

$$Q = I_3 + \frac{1}{2}Y + \frac{1}{3}D \quad (13)$$

$D$  being the so-called supercharge [2]. If (12) applies, i.e., if strong and electromagnetic interactions conserve  $Y$  and  $D$  separately, simul-



taneous reality of  $c_0$ ,  $c_1$ ,  $c'_1$  ensures that weak interactions conserve  $CP$  and  $T$ .

However, no long-lived supercharged ( $D \neq 0$ ) particles have been found so far, and it appears therefore more likely that semi-strong interactions violate  $Y$  and  $D$  conservation while maintaining conservation of the "effective hypercharge"

$$Y' = Y + \frac{2}{3}D \quad (14)$$

Under these conditions, all supercharged particles including the trions themselves can be strong interaction resonances among known long-lived baryons and mesons. The implication for weak interactions would be interesting. The constants  $c_0$ ,  $c_1$  being made real as before,  $c'_1$  retains a phase which cannot be reduced to zero, and interference between the two terms in (10) implies  $CP$  and  $T$  violation in  $|\Delta Y| = 1$  transitions, of course with conservation of  $CPT$ . This violation must be weak for decays of ordinary ( $D = 0$ ) particles, because its magnitude is given by

$$|c'_1/c_1|R,$$

with  $R$  the strength ratio of semi-strong to strong interactions, and we have seen that  $|c'_1/c_1|$  is expected to be small.

In contrast with the case  $|\Delta Y| = 1$ ,  $CP$  and  $T$  are separately conserved for  $\Delta Y = 0$  transitions in the scheme just described.

#### ARE SYMMETRIES FUNDAMENTAL?

In the discussion given above for electromagnetic and weak interactions the  $C$ ,  $P$  and  $T$  symmetry properties of these interactions were not assumed a priori but rather appeared as consequences of their explicit form in terms of the basic triplet fields. Also for strong interactions the question arises whether they could be given a basically simple algebraic form embodying not only their symmetries like isospin,  $SU(3)$ ,  $SU(6)$  (or higher symmetries for trions), but also the violations of all these symmetries except isospin. To assume a very strong interaction term of very high symmetry and some symmetry violating terms of medium strength is highly unattractive when dealing with the basic equations themselves. To derive unambiguously all the known  $SU(3)$  and  $SU(6)$  regularities and violations from basic equa-

tions which do not contain these symmetries a priori in some way, looks like a difficult task as long as one is forced to deal with untractable strong coupling equations by applying radical approximations which are tailored to the necessity of reproducing the very same properties one wants to explain. The basic trouble is of course that no "hydrogen atom" or no "dilute gas" have yet been found in the realm of strong interactions. Are there any places left where we could look with some hope of success for such basically simple configurations? From what we know, there is not much hope at low or moderate energies, nor is the situation likely to be better at very high energy as long as momentum transfers are small or moderate. But hope remains for the region where both energy and momentum transfers are very large compared to the proton mass. Whether triplets exist or not, it will probably be very instructive to explore the region

$$s \gtrsim (5 \text{ GeV}/c)^2, \quad -t \gtrsim (5 \text{ GeV}/c)^2.$$

Strong interaction cross-sections may drop below  $10^{-35} \text{ cm}^2$ , but the rewards could be great.

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# THE CHARGE CONJUGATION OPERATION AND MIXED SPACE-TIME-INTERNAL SYMMETRY GROUPS

S. OKUBO and R. E. MARSHAK

*University of Rochester, Rochester, N.Y.*

*(Received June 14, 1965)*

As the number of elementary particles has increased, attempts have multiplied to enlarge the underlying group structure governing the interactions of these particles. Except for Wigner's supermultiplet theory for atomic nuclei [which involved a mixing of the  $SU(2)$  spin group with the  $SU(2)$  isospin group into the  $SU(4)$  group], these attempts have until quite recently maintained a clear distinction between the space-time groups and the internal symmetry groups. Under the internal symmetry groups, we would list the following:

1.  $SU(3)$  or  $SU(2)$ +charge ( $Q$ ) and hypercharge ( $Y$ ) gauge groups
2. Baryon gauge group ( $B$ )
3. Lepton gauge group ( $L$ ) (there may be two lepton groups)
4. Charge conjugation operation ( $C$ )
5. Permutation group for statistics (Bose-Einstein or Fermi-Dirac).

Actually, it is not completely true that the internal symmetry groups are unrelated to the Poincaré group. For example, the following empirical relation holds [1] between the intrinsic spin  $J$  of a hadron particle and its baryon number:

$$2J + B = 0 \pmod{2}.$$

A similar relation holds for a lepton particle when the lepton number  $L$  replaces  $B$  in Eq. (1). There is also the relation between spin and statistics (permutation group) which can be derived from first principles in quantum field theory. However, the charge conjugation operation  $C$  seems to play a particularly interesting role in relating the space-time and internal symmetry groups.

First, there is the  $TCP$  theorem which connects  $C$  to the discrete Lorentz transformations ( $T$  and  $P$ ) on the basis of some general

causality arguments and the hypothesis of local interactions. It is remarkable that one can draw conclusions about  $C$  invariance from a knowledge of the space-time invariances  $T$  and  $P$ . Secondly,  $C$  possesses the property of not commuting with any of the gauge groups ( $Q$ ,  $Y$ ,  $B$ , or  $L$ ); indeed,  $C$  changes the signs of all these quantum numbers. Since  $Q$  and  $Y$  are related to  $I_3$  (through the Gell-Mann-Nishijima relation  $Q = I_3 + \frac{1}{2}Y$ ),  $I_3$  must also change sign under  $C$ . This implies that  $C$  does not commute with the isospin group  $SU_2^{(I)}$ . Since  $SU(3)$  contains  $SU_2^{(I)}$  and  $Y$  as subgroups, the  $C$  operation does not commute with  $SU(3)$  although it preserves the Lie algebra [2] of  $SU(3)$ . Finally, the  $C$  operation is involved in "crossing symmetry" wherein the analytic properties of the amplitude for, say, the scattering process  $P_1 + P_2 \rightarrow P_3 + P_4$  are related to the amplitude for the scattering process  $P_1 + \bar{P}_3 \rightarrow \bar{P}_2 + P_4$  (where  $\bar{P}$  is the anti-particle of  $P$ ).

Since the charge conjugation operation appears to be so intimately involved with many of the internal symmetry groups and with some of the space-time groups, it is of interest to examine the charge conjugation properties of the recently developed higher symmetry groups [3] which mix the space-time and internal symmetry groups. A paper by one of the authors [4] has shown that the charge conjugation operation in its usual formulation is not consistent with invariance under the  $\tilde{U}(12)$  group, which has been put forward [5] as a satisfactory relativistic generalization of  $SU(6)$  which mixes spin and unitary spin. In this "Prelude", dedicated to Professor V. F. Weisskopf, we restate the argument in a form which should appeal to the man we are honoring and comment further on how the difficulty can be remedied for a mixed space-time-internal symmetry group like  $\tilde{U}(12)$ .

The argument which demonstrates the inconsistency between the usual definition of the charge conjugation operation for the basic Dirac spinor fields which are used to define the multispinor representations of the  $\tilde{U}(12)$  group and the  $\tilde{U}(12)$ -invariant definition of, say, the baryon-meson vertex proceeds as follows. Let us write down the baryon-meson vertex according to the  $\tilde{U}(12)$  prescription, namely:

$$\Lambda(p, p') = F(q^2) \bar{\Psi}^{ABC}(p) \Psi_{ABD}(p') M_C^D(q) \quad (1)$$

where  $\Psi_{ABC}$  is the 364-dimensional baryon multispinor and  $M_A^B$  the 143-dimensional meson multispinor;  $\Psi_{ABC}$  transforms as  $u_A \cdot u_B \cdot u_C$



and  $M_A^B$  as  $\bar{u}^B u_A$  where  $u_A$  ( $A = 1, \dots, 12$ ) is a 12-component spinor representing an SU(3) triplet of 4-component Dirac quarks.  $F(q^2)$  in Eq. (1) is the form factor of the vertex with momentum transfer  $q = p' - p$ .

On the other hand, the anti-baryon-meson vertex must take the form:

$$\bar{\Lambda}(p, p') = G(q^2) \bar{\Phi}^{ABD}(p') \Phi_{ABC}(p) M_D^C(q) \quad (2)$$

where  $\Phi_{ABC}$  is the anti-baryon multispinor corresponding to the baryon multispinor  $\Psi_{ABC}$ . Now the charge conjugation transform  $v$  of the Dirac spinor  $u$  is defined in the usual way by:

$$v = C\bar{u}^T, \quad \bar{v} = -u^T C^{-1} \quad (3)$$

with  $C$  satisfying the conditions:

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu, \quad C^T = -C, \quad C^\dagger C = 1. \quad (3a)$$

Hence the anti-baryon multispinors transform according to the rules:

$$\begin{aligned} \Phi_{ABC}(p) &= C_{A'}^A \cdot C_{B'}^B \cdot C_{C'}^C \bar{\Psi}^{A'B'C'}(p) \\ \bar{\Phi}^{ABC}(p) &= (C^{-1})_A^{A'} \cdot (C^{-1})_B^{B'} \cdot (C^{-1})_C^{C'} \Psi_{A'B'C'}(p). \end{aligned} \quad (4)$$

If we insert the expressions given by Eq. (4) into Eq. (2), we obtain the result:

$$\bar{\Lambda}(p, p') = G(q^2) \bar{\Psi}^{ABC}(p) \Psi_{ABD}(p') \cdot C_C^{C'} \cdot (C^{-1})_{D'}^D \cdot M_{D'}^{C'}. \quad (5)$$

Eq. (5) would have the same  $\tilde{U}(12)$ -invariant form as Eq. (1) if the following relation were true:

$$C_C^{C'} M_{D'}^{C'} (C^{-1})_{D'}^D = M_D^D \quad (6)$$

or more succinctly (using the antisymmetric properties of  $C$ ):

$$CMC^{-1} = M^T. \quad (6a)$$

Unfortunately, the meson multispinor contains pseudoscalar and vector mesons with opposite  $C$  parity and hence Eq. (6a) is not true. It follows that the usual charge conjugation operation  $C$  is a Lorentz-invariant but not a  $\tilde{U}(12)$ -invariant concept. If we work with the conventional definition of the charge conjugation operation and insist on maintaining  $\tilde{U}(12)$  invariance, we would be compelled to assert that

the baryon-meson vertex must vanish. [We emphasize that the *anti-baryon-antimeson* vertex is a  $\tilde{U}(12)$ -invariant concept even with the conventional definition of  $C$ .]

The difficulty of reconciling the usual charge conjugation operation  $\tilde{U}(12)$  symmetry can be seen in a more general way as follows. Invariance under the  $\tilde{U}(12)$  group requires that any effective matrix element is invariant under the transformation:

$$\begin{aligned} u(p) &\rightarrow Su(p), & \bar{u}(p) &\rightarrow \bar{u}(p)S^{-1} \\ v(p) &\rightarrow Sv(p), & \bar{v}(p) &\rightarrow \bar{v}(p)S^{-1} \end{aligned} \quad (7)$$

where  $u$  and its charge conjugate transform  $v$  are the 12-component spinors (with  $u$  and  $v$  transforming as covariant vectors according to Salam et al. [6]) and  $S$  is an arbitrary  $12 \times 12$  matrix satisfying the condition:

$$S^\dagger \cdot \gamma_4 \cdot S = \gamma_4. \quad (8)$$

Condition (8) ensures the invariance of the mass term in the  $\tilde{U}(12)$  theory. However, the usual definition of the charge conjugation transform given by Eq. (3) is consistent with the  $S$  matrix defined by (8) only if it satisfies the additional constraint:

$$SCS^T = C. \quad (9)$$

Since the condition (9) implies that  $S$  is a symplectic matrix, it follows that Eq. (3) can only be invariant under the  $\tilde{Sp}(12)$  subgroup of  $\tilde{U}(12)$  and not under the *complete* group  $\tilde{U}(12)$ .

Even if we allow the charge conjugation transform spinor,  $v$ , to be a contravariant vector in contrast to the covariant vector  $u$ , we still cannot reconcile  $\tilde{U}(12)$  invariance with the usual charge conjugation operation. In this case, Eq. (7) changes into:

$$\begin{aligned} u(p) &\rightarrow Su(p), & \bar{u}(p) &\rightarrow \bar{u}(p) \cdot S^{-1} \\ v(p) &\rightarrow (S^{-1})^T v(p), & \bar{v}(p) &\rightarrow \bar{v}(p) \cdot S^T \end{aligned} \quad (10)$$

and Eq. (9) into:

$$SC = CS. \quad (11)$$

The inconsistency now follows from the fact that condition (11) can not even be satisfied for an arbitrary matrix  $S$  corresponding to a pure Lorentz transformation.

One might inquire whether there is any way of reconciling the  $\tilde{U}(12)$  group with the charge conjugation operation. One possible way is to use the Klein-Gordon equation for the basic spinors out of which the baryon and meson multispinors are constructed. That is to say, the basic quark spinor  $u_A$  would satisfy the Klein-Gordon equation [7]:

$$(p_\mu^2 + m^2)u_A = 0 \quad (A = 1, \dots, 12) \quad (12)$$

instead of the Dirac equation:

$$(i\gamma_\mu \cdot p_\mu + m)u_A = 0. \quad (13)$$

Since Eq. (12), in contrast to Eq. (13), does not contain any  $\gamma_\mu$  matrices, it is invariant under  $\tilde{U}(12)$  while Eq. (13) is not. Unfortunately, Eq. (12) possesses twice as many solutions as Eq. (13), and we would have trouble interpreting the redundant solutions for baryons; indeed, this doubling reflects the fact that the Klein-Gordon equation is not really irreducible with respect to the Poincaré group. We may overcome this objection by working with the Klein-Gordon equation for a 6-component spinor  $\psi_a$  ( $a = 1, \dots, 6$ ), namely:

$$(p_\mu^2 + m^2)\psi_a = 0 \quad (a = 1, \dots, 6) \quad (14)$$

which is actually invariant under  $SL(6, C)$  [rather than  $\tilde{U}(12)$ ]. We may also write down an  $SL(6, C)$ -invariant definition of charge conjugation, namely:

$$\phi^a = \psi_a^\dagger. \quad (15)$$

However, if we try to use an  $SL(6, C)$ -invariant definition of parity [7] as well, i.e.

$$\psi_a(\mathbf{x}, t) = \pm \psi_a(-\mathbf{x}, t) \quad (16)$$

and require parity conservation for the strong and electromagnetic interactions, then the cross-section, say, for electron-nucleon scattering turns out to be very different from that given by the well-confirmed Rosenbluth formula. Hence, the Klein-Gordon equation is not a way out of our dilemma.

There is, however, a way to reconcile  $\tilde{U}(12)$  with the charge conjugation operation. This may be achieved by modifying the transformation properties of the Dirac spinors  $u$  and  $v$  under  $\tilde{U}(12)$  as

follows [ $S$  satisfies condition (8)]:

$$\begin{aligned} u(p) &\rightarrow u'(p) = Su(p) \\ v(p) &\rightarrow v'(p) = C(S^{-1})^T C^{-1} v(p) \end{aligned} \quad (17)$$

instead of Eq. (7) or (10). Then, as has been remarked already by some authors [8], one can maintain the relation:

$$\begin{aligned} v(p) &= C \cdot \bar{u}(p) \\ v'(p) &= C \cdot \bar{u}'(p). \end{aligned} \quad (18)$$

Hence, we can avoid a contradiction with  $\tilde{U}(12)$  if we are willing to forego the assignment of a simple tensor character (covariant or contravariant) simultaneously to both  $u$  and  $v$ .

The fact that the transformation property of  $v$  in Eq. (17) is asymmetrical compared to that of  $u$  does not seem to be too steep a price to pay for saving the consistency of  $\tilde{U}(12)$  with charge conjugation. For, let us recall [2] that the charge conjugation operation in any symmetry group may be best defined to be an involutory outer automorphism of the group. In connection with the  $\tilde{U}(12)$  group, let us denote by  $Q_A$  ( $A = 1, \dots, 144$ ), its 144 defining generators, i.e.  $Q_A = \lambda_i, \lambda_i \gamma_5, \lambda_i \gamma_\mu, \lambda_i \gamma_5 \gamma_\mu, \lambda_i \sigma_{\mu\nu}$  ( $i = 1, \dots, 9; \mu, \nu = 1, \dots, 4$ ). Then, these generators satisfy the Lie algebra:

$$[Q_A, Q_B] = C_{AB}^D Q_D. \quad (19)$$

We now note that a mapping on this algebra defined by

$$Q_A \rightarrow \sigma(Q_A) = -C Q_A^T C^{-1} \quad (20)$$

is an involutory automorphism, [9] since we have

$$[\sigma(Q_A), \sigma(Q_B)] = C_{AB}^D \sigma(Q_D) \quad (21a)$$

$$\sigma^2 = 1. \quad (21b)$$

Actually, this mapping is an outer automorphism [10]. Eq. (20) will be used as the definition of the charge conjugation operation.

We next extend the mapping  $\sigma$  defined above to the entire  $\tilde{U}(12)$  group as follows. Any  $\tilde{U}(12)$  transformation matrix  $S$  may be written as

$$S = \exp [i\theta_A \cdot Q_A] \quad (22)$$



where  $\theta_A$  ( $A = 1, \dots, 144$ ) are parameters. Then, we define  $\sigma(S)$  by

$$\sigma(S) = \exp [i\theta_A \cdot \sigma(Q_A)] \quad (23)$$

or

$$\sigma(S) = C \cdot (S^{-1})^T \cdot C^{-1}.$$

Thus, if  $u$  transforms in accordance with:  $u(p) \rightarrow u'(p) = Su(p)$  then its charge conjugate spinor  $v$  must transform as  $v(p) \rightarrow v'(p) = \sigma(S)v(p)$  which is the same as Eq. (17). Thus, we have justified Eq. (17) on the basis of the general group-theoretic definition of charge conjugation. It should be emphasized that one must be careful in calculations involving anti-particle states in the  $\tilde{U}(12)$  theory, since  $v$  no longer possesses a simple transformation property in  $\tilde{U}(12)$ .

We acknowledge useful conversations with Drs. L. K. Pandit, Riazuddin and C. Ryan. Also, we are grateful to Professor E. C. G. Sudarshan for a valuable remark. This work was partially supported by U.S. Atomic Energy Commission.

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$$u_A = \begin{pmatrix} \psi_a \\ \frac{i}{m} \cdot \sigma_\mu \bar{p}_\mu \psi_a \end{pmatrix}$$

and define the parity operation by  $\psi_a(\mathbf{x}, t) \rightarrow (1/m)\sigma_\mu p_\mu \psi_a(-\mathbf{x}, t)$  [cf. L. M. Brown, Phys. Rev. **111** (1958) 957]. The Rosenbluth formula would follow from these definitions but the  $\sigma_\mu$  in the parity operation destroys the  $SL(6, C)$  invariance (cf. below).

- 8) J. S. Bell and H. Ruegg; CERN preprint;  
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- 9) Conditions (19) and (21a) define an automorphism of the group, (21b) makes it involutory.
- 10) That  $\sigma$  is an outer automorphism can be proved as follows. Suppose that  $\sigma$  were an inner automorphism. Then a  $12 \times 12$  matrix  $U$  must exist satisfying the condition:  $UQ_A U^{-1} = \sigma(Q_A) = -CQ_A^T C^{-1}$ . When one chooses  $Q_A = \gamma_\mu$  and  $\gamma_5$ , we must have  $U\gamma_\mu U^{-1} = +\gamma_\mu$ ,  $U\gamma_5 U^{-1} = -\gamma_5$  which are in contradiction with one another. Hence  $\sigma$  must be an outer automorphism.

# GIANT RESONANCES IN NUCLEI

J. D. WALECKA \*

*Department of Physics, Institute of Theoretical Physics  
Stanford University, Stanford, California*

(Received April 28, 1965)

In this note we would like to make some very simple observations concerning Giant Resonances in nuclei. Most of this material is well known, in one context or another, and this is merely an attempt to try and tie things together, and perhaps gain a little insight in so doing.

If  $S_{JM}$  is a multipole operator, and  $|G\rangle$  is the nuclear ground state, then one has as an identity

$$\sum_M \langle G | [S_{JM}^\dagger [H, S_{JM}]] | G \rangle = 2 \sum_M \sum_n (E_n - E_0) |\langle n | S_{JM} | G \rangle|^2. \quad (1)$$

If we take  $S_{1M} = \sum_{i=1}^A \tau_3(i) x_{1M}(i)$  (the dipole moment of the charge density) and assume that the interaction Hamiltonian  $V$  commutes with  $S_{1M}$ , then we find

$$2 \sum_M \sum_n (E_n - E_0) |\langle n | \sum_{i=1}^A \tau_3(i) x_{1M}(i) | G \rangle|^2 = \frac{3\hbar^2 A}{m} \quad (2)$$

which is the familiar Thomas-Reiche-Kuhn Dipole Sum Rule. If we attempt to go a step farther and take  $S_{JM} = \sum_{i=1}^A \tau_3(i) [\sigma(i) \odot x(i)]_{JM}$  and again assume  $[V, S_{JM}] = 0$ , then we have

$$2 \sum_M \sum_n (E_n - E_0) |\langle n | \sum_{i=1}^A \tau_3(i) [\sigma(i) \odot x(i)]_{JM} | G \rangle|^2 = (2J+1) \frac{\hbar^2 A}{m} \quad (3)$$

where we have assumed  $|G\rangle$  has spin zero. Summing over  $J$  gives

$$2 \sum_M \sum_\lambda \sum_n (E_n - E_0) |\langle n | \sum_{i=1}^A \tau_3(i) \sigma_\lambda(i) x_M(i) | G \rangle|^2 = 3 \cdot \frac{3\hbar^2 A}{m}. \quad (4)$$

This result is independent of the properties of  $|G\rangle$ .

\* A. P. Sloan Foundation Fellow.

\*\*  $[\sigma \odot x]_{JM} = \sum_{qq'} (1q1q' | 1JM) \sigma_{1q} x_{1q'}$ . In general we use the notation of Edmonds [1].

Now it is known experimentally, from photoabsorption cross sections, that most of the electric dipole strength in nuclei is systematically concentrated in the Giant Electric Dipole Resonance which falls in the region 15–25 MeV for the heaviest to the lightest nuclei. This resonance exhausts most of the sum rule in Eq. (1), although it falls to about one-half of the sum rule in the lightest nuclei. This question arises as to whether or not a similar situation exists with respect to the operator in Eq. (3), and if it does, how would this be manifested experimentally? The simplest model of the Giant Electric Dipole Resonance is that due to Goldhaber and Teller [2]. Here the protons are assumed to move as a unit against the neutrons. This creates a large displacement of the charge from the center of mass, and hence a very large electric dipole moment. In fact this mode of motion exhausts the dipole sum rule as can be seen as follows: Imagine that the displacement is governed by the simple Hamiltonian  $H = p^2/2\mu + \frac{1}{2}\mu\omega^2 q^2$  where  $q$  is the relative coordinate of the center of mass of the neutrons and protons, and  $\mu = \frac{1}{4}mA$  (assuming  $N = Z$ ). If we quantize this Hamiltonian and chose  $\hbar\omega$  to be the Giant Resonance energy, then we can very simply calculate the transition matrix elements of the operator

$$\sum_{i=1}^A \tau_3(i)x(i) \rightarrow \int \rho_V(x)x dx = \int x dx [\rho_0(|x - \frac{1}{2}q|) - \rho_0(|x + \frac{1}{2}q|)] \quad (5)$$

where  $\rho_0$  is the proton charge density. Expanding for small  $q$ 's we find [3]

$$2\hbar\omega |\langle 1^- | \sum_{i=1}^A \tau_3(i)x(i) | G \rangle|^2 = \frac{3\hbar^2}{\mu} Z^2 = \frac{3\hbar^2 A}{m} \quad (6)$$

which is the sum rule value. Let us now assume the approximate spin and isotopic spin independence of the nucleon-nucleon force. In this case, since the nucleus is made up of four kinds of particles, neutrons with spin up and down, and protons with spin up and down, one expects to see oscillations which are degenerate with the Giant Electric Dipole Resonance (since the restoring force is the same) in which protons with spin up and neutrons with spin down move against protons with spin down and neutrons with spin up \*. One might expect

\* Similar oscillations have been considered by Fallieros, Ferrell, and Pal [4] and Glassgold, Heckrotte, and Watson [5].



these oscillations to exhaust the sum rules of Eq. (2). This can be seen in the simple "semi-classical" model of these oscillations where we compute [6]

$$\begin{aligned} \sum_{i=1}^A \tau_3(i)[x(i) \odot \sigma(i)]_{JM} &\rightarrow \sqrt{\frac{4\pi}{3}} \int \mu_V(\mathbf{x}) \cdot \mathbf{x} Y_{J11}^M(\Omega_x) d\mathbf{x} \\ &= \sqrt{\frac{4\pi}{3}} \int \mathbf{x} Y_{J11}^M(\Omega_x) \cdot [\rho_0(|\mathbf{x} - \frac{1}{2}\mathbf{q}|)\sigma(1) + \rho_0(|\mathbf{x} + \frac{1}{2}\mathbf{q}|)\sigma(2)] d\mathbf{x}. \end{aligned} \quad (7)$$

If we take the ground state to be  $S = 0$  and the excited state to be  $S = 1$ , then we find, again expanding for small  $\mathbf{q}$

$$2\hbar\omega |\langle J^\pi || \sum_{i=1}^A \tau_3(i)[\sigma(i) \odot x(i)]_J || G \rangle|^2 = (2J+1) \frac{\hbar^2 Z^2}{\mu} = (2J+1) \frac{\hbar^2 A}{m} \quad (8)$$

which is the sum rule value of Eq. (3).

The results of these simple models can be seen to hold very generally by using the powerful techniques of Group Theory [7, 8]. Suppose one defines the 15 operators

$$\begin{aligned} T^\alpha &\equiv \frac{1}{2} \sum_{i=1}^A \tau^\alpha(i) & S_\lambda &\equiv \frac{1}{2} \sum_{i=1}^A \sigma_\lambda(i) \\ Y_\lambda^\alpha &\equiv \frac{1}{2} \sum_{i=1}^A \tau^\alpha(i) \sigma_\lambda(i) & \alpha, \lambda &= 1, 2, 3, \end{aligned} \quad (9)$$

then the commutator of any two of these  $G^{(\alpha)}$  ( $\alpha = 1 \dots 15$ ) is again a  $G^{(\alpha)}$ . These operators are traceless and Hermitian. If we now define  $\phi_i \equiv \eta_{m_s} \zeta_{m_t}$ ,  $m_s = \pm \frac{1}{2}$ ,  $m_t = \pm \frac{1}{2}$  [ $i = 1 \dots 4$ ] where  $\eta_{m_s}$  and  $\zeta_{m_t}$  are Pauli spinors, then the set of transformations defined by

$$R(\omega) \phi_i \equiv [e^{i\omega_\alpha G_\alpha}]_{ji} \phi_j \quad (10)$$

where  $\omega_\alpha$  are real numbers, form a 15 parameter Lie Group, SU(4), the group of  $4 \times 4$  unitary unimodular matrices. If the nucleon-nucleon force is independent of spin and isotopic spin then \*

$$[G^{(\alpha)}, H] = 0 \quad \alpha = 1 \dots 15 \quad (11)$$

\* This eliminates spin and isospin dependent forces, but leaves us with Wigner and Majorana forces.

and the degenerate eigenstates of  $H$  form a basis for an irreducible representation of  $SU(4)$ . This may not be such a bad approximation since a Serber force with only a weak spin dependence can fit nucleon-nucleon scattering up to 90 MeV or so; however, it certainly makes a convenient starting point. The Giant Electric Dipole Resonance, since it exhausts the sum rule, can be thought of very crudely as the state  $\sum_{i=1}^A \tau_3(i)x(i)|G\rangle$ . We will limit our considerations to nuclei of the type  $A = 4n$  for which the ground state may be expected to belong to the identity representation of  $SU(4)$  [this leads to the most symmetric spatial state and hence to the maximum overlap of the nucleon wave functions]. Thus the above form *suggests* that we assign the Giant Electric Dipole resonance to the  $(2, 1, 1)$ , or 15 dimensional representation of  $SU(4)$ . If we further assign  $L = 1$  to these resonances, we then expect to find all of the resonances of Table 1.

TABLE 1

$L$	$S$	$J^\pi$	$T$
1	0	$1^-$	1
1	1	$0^-1^-2^-$	0
1	1	$0^-1^-2^-$	1

Thus  $SU(4)$  tells us we should see *all of these giant resonances as degenerate states*. One can make an even stronger statement using  $SU(4)$ . Consider all the states of a given energy  $(E_n, m)$  where  $m$  labels all the other quantum numbers. Now introduce the following 3 operators

$$U^1 \equiv Y_\lambda^1 \quad U^2 \equiv Y_\lambda^2 \quad U^3 \equiv T^3. \quad (12)$$

The  $U^i$  form an  $SU(2)$  subgroup of  $SU(4)$ .

$$[U^i, U^j] = i\epsilon_{ijk} U^k. \quad (13)$$

If we define  $U_- \equiv U^1 - iU^2$ , and let  $\omega(i)$  be an arbitrary function of  $x(i)$  then

$$[U_-, \sum_{i=1}^A \tau_3(i)\omega(i)] = 2 \sum_{i=1}^A \tau_-(i)\omega(i)\sigma_\lambda(i). \quad (14)$$

Taking matrix elements of this expression, and using the fact that for each  $E_n$  we can label the states by  $(U, U_3)$  since  $U$ -spin is just a subgroup of  $SU(4)$ , we find [8]

$$\frac{1}{2} \sum_m |\langle E_n m | \sum_{i=1}^A \tau_3(i) \omega(i) | G \rangle|^2 = \sum_m |\langle E_n m | \sum_{i=1}^A \tau_{-}(i) \sigma_{\lambda}(i) \omega(i) | G \rangle|^2. \quad (15)$$

Using the isotopic spin subgroup and repeating these arguments leads to

$$\sum_m |\langle E_n m | \sum_{i=1}^A \tau_3(i) \omega(i) | G \rangle|^2 = \sum_m |\langle E_n m | \sum_{i=1}^A \tau_3(i) \sigma_{\lambda}(i) \omega(i) | G \rangle|^2. \quad (16)$$

With  $SU(4)$  invariance we find that the matrix elements of the operators in Eq. (16) must be equal at all energies. Thus *if experimentally there is a resonance in the matrix element on the left hand side of (16), there must necessarily be one in the matrix element on the right hand side.*

It is interesting to see how this comes out of a more sophisticated model of the Giant Resonance. We make a particle-hole model of the resonance, and expand

$$|n\rangle = \sum_{\alpha\beta} C_{\alpha\beta}^* a_{\alpha}^{\dagger} b_{\beta}^{\dagger} |G\rangle \quad (17)$$

where  $\alpha = (n, l, m_l, \frac{1}{2}, m_s, \frac{1}{2}, m_t) \equiv (a, m_l, m_s, m_t)$ , and  $a^{\dagger}$  creates a particle,  $b^{\dagger}$  a hole. Linearizing the equations of motion in the familiar Tamm-Damcoff approximation leads to the set of equations for the coefficients  $C_{\alpha\beta}$  [9, 10, 11]

$$(\varepsilon_{\alpha} - \varepsilon_{\beta} - \omega) C_{\alpha\beta} + \sum_{\rho\mu} \{ [\langle \rho - \beta | V | -\mu \alpha \rangle - \langle \rho - \beta | V | \alpha - \mu \rangle] S_{-\mu} S_{-\beta} \} C_{\rho\mu} = 0 \quad (18)$$

where  $-\alpha \equiv (a, -m_l, -m_s, -m_t)$  and  $S_{\alpha} \equiv (-1)^{l-m_l+\frac{1}{2}-m_s+\frac{1}{2}-m_t}$ . If we assume that  $V$  is independent of spin and isotopic spin, and that  $\varepsilon_{\alpha} = \varepsilon_{n_a l_a} \equiv \varepsilon_a$  we find after some algebra

$$(\varepsilon_a - \varepsilon_b - \omega) C^{TSL}(ab) + \sum_{lm} v_{ab, lm}^{TSL} C^{TSL}(lm) = 0 \quad (19)$$

where  $(ab)$  and  $(lm)$  label the particle-hole states involved, and

$$v_{ab, lm}^{TSL} \equiv - \sum_{L'} (2L'+1) \begin{pmatrix} l_l & l_b & L' \\ l_a & l_m & L \end{pmatrix} \times \\ \times [\langle lbL' | V | amL' \rangle - 4\delta_{S0} \delta_{T0} (-1)^{l_a+l_m-L'} \langle lbL' | V | maL' \rangle]. \quad (20)$$

Thus the 15 states with  $(T = 1, S = 1)$ ,  $(T = 1, S = 0)$ ,  $(T = 0, S = 1)$  and a given  $LM_L$  are degenerate while the interaction separates the state with  $(S = 0, T = 0)$ . This is easily understood from the point of view of  $SU(4)$ . A particle belongs to the representation (4) while a hole belongs to the adjoint representation  $\bar{(4)}$ . The particle-hole states are the direct product states which are reduced by the rule  $4 \otimes \bar{4} = 1 \oplus 15$ . Thus again we see that the Giant Resonances belong to the (15) dimensional representation of  $SU(4)$ .<sup>†</sup>

The above equations can be reduced to very simple form in the case of an interaction  $V = -a\delta(r_1 - r_2)$  for then one finds, by carrying out the angular momentum sums

$$v_{ab;lm}^{(15)L} = \xi v_{ab}^L v_{lm}^L$$

$$v_{ab}^L \equiv (-1)^{l_a} \begin{pmatrix} l_a & L & l_b \\ 0 & 0 & 0 \end{pmatrix} \sqrt{(2l_a+1)(2l_b+1)} \quad (21)$$

where

$$\xi = 2aI = \frac{a}{4} \int R_{n_a l_a} R_{n_b l_b} R_{n_l l_l} R_{n_m l_m} r^2 dr. \quad (22)$$

If we assume  $I$  to be a constant, then the potential is separable and the eigenvalue equation is simply

$$\frac{1}{\xi} = \sum_{ab} \frac{(v_{ab}^L)^2}{\omega - \epsilon_{ab}}. \quad (23)$$

Analysis of this equation shows that with an *attractive interaction* ( $\xi > 0$ ) one level is *pushed up in energy* and collects most of the electric dipole strength. This observation is due to Brown and Bolsterli [12]. This state with a given  $L$  is then combined with the spin-isospin states corresponding to the (15) dimensional representation of  $SU(4)$  to form the supermultiplet of Giant Resonances.

One can ask to what extent these considerations are modified by the inclusion of spin effects. The effect of spin orbit splittings in the configuration energies and of a spin dependence in the nucleon-nucleon force fit to low energy scattering has been investigated by Lewis and Walecka [13, 14], and deForest [15] in  $C^{12}$  and  $O^{16}$ . The Giant Resonance states  $S = 1$ ,  $T = 1$ ,  $J^\pi = 0^-, 1^-, 2^-$  are

<sup>†</sup> These results also hold in the Random Phase Approximation [9, 10, 11] for the Giant Resonance states.



shifted somewhat from the  $S = 0, T = 1, J^\pi = 1^-$  state but still occur in the region 20–25 MeV, and one state still carries a majority of the corresponding multiple strength. This is in essential agreement with the earlier calculation of Brown and co-workers [16, 17].

Finally, we consider the question of where these Giant Resonances would manifest themselves experimentally. It is clear, from the simple Goldhaber-Teller model that the spin-isospin oscillations never develop a large charge dipole moment since if the protons with spin up move against the protons with spin down, the center of mass and center of charge remain the same. Hence they will not give rise to large photon cross sections, and one must look elsewhere. One place to look is in muon capture. The muon capture rate can be written as [7]

$$A_{\mu c}(n \leftarrow G) = \frac{m_\mu |\phi_{\mu \text{av}}|^2}{2\pi} [G_V^2 (M_V^2)_{n0} + 3G_A^2 (M_A^2)_{n0} + (G_P^2 - 2G_P G_A) (M_P^2)_{n0}] + A'_{\mu c} \quad (24)$$

where  $A'_{\mu c}$  contains the nucleon recoil corrections ( $\approx (v/c)_{\text{nucleon}}$ ),  $|\phi_{\mu \text{av}}|^2$  is the average of the square of the bound state muon wave function over the nucleus,  $G_{V, A, P}$  are coupling constants, and

$$(M_{V, A, P}^2)_{n0} = \left( \frac{v_{n0}}{m_\mu} \right)^2 \sum_m \int \frac{d\hat{v}}{4\pi} |\langle E_n m | \sum_{i=1}^A \hat{O}_{V, A, P}(i) e^{-i\mathbf{v}_{n0} \cdot \mathbf{x}(i)} | G \rangle|^2 \quad (25)$$

$$\hat{O}_V = \tau_- \quad \hat{O}_A = \frac{1}{\sqrt{3}} \tau_- \sigma \quad \hat{O}_P = \tau_- \sigma \cdot \hat{v}.$$

$v_{n0} = v_{\text{max}} - (E_n - E_0)$  is the neutrino momentum corresponding to an excitation of the nucleus to the state  $|n\rangle$ . If  $|G\rangle$  is a doubly magic nucleus, or belongs to the identity representation of SU(4) then the allowed capture vanishes as  $\sum_{i=1}^A \tau_-(i) |G\rangle = \sum_{i=1}^A \tau_-(i) \sigma_z(i) |G\rangle = 0$ . Therefore, the leading term in the capture rate is first forbidden, that is, proportional to  $\mathbf{v}_{n0} \cdot \mathbf{x}(i)$ , and the matrix elements are just those we've been considering.\* If the nucleon-nucleon force is spin and iso-

\* Even if we don't make an expansion of the exponential, the matrix element  $M_V^2$  can still be evaluated explicitly in the Goldhaber-Teller model and one finds

$$|\langle 1^- | \sum_{i=1}^A \tau_3(i) j_1(qx_i) Y_1(\mathcal{Q}_i) | G \rangle|^2 = \left( \frac{\hbar^2 q^2}{2\mu} \right) \frac{1}{\hbar\omega} |M_0(q)|^2$$

spin independent then our previous considerations tell us  $(M_V^2)_{n0} = (M_A^2)_{n0} = (M_P^2)_{n0}$ . Since we know that  $(M_V^2)_{n0}$  is dominated by the Giant Electric Dipole resonance, this says that muon capture should take place predominantly through the degenerate set of Giant Resonances. Foldy and Walecka have used this observation to calculate total muon capture rates in  $\text{Ca}^{40}$ ,  $\text{O}^{16}$ ,  $\text{C}^{12}$ , and  $\text{He}^4$  and the overall agreement is very good [8]. This indicates that the axial vector strength is distributed the same way as the vector strength in nuclei and is strong, though rather indirect, evidence for the other Giant Resonances.

A more direct way to look for these states is to make use of the fact that the operators we are considering are closely related to the transverse electromagnetic multipole operators discussed in Blatt and Weisskopf [18]. With real photons one sees only the long wave length parts of each multipole operator. With electrons, the same transverse electromagnetic multipole operators (in addition to the Coulomb multipoles) govern the cross section. Here, however, the relevant wave number in the multipoles is  $\mathbf{q} = \mathbf{K}_f - \mathbf{K}_i$ , the three momentum transferred by the electron to the nucleus, while exciting the nucleus to a state of definite energy.  $|\mathbf{q}|$  can be varied through any value  $|\mathbf{q}| \geq \Delta E_n$ , while with real photons one is limited to just one value,  $|\mathbf{q}| = K_\gamma = \Delta E_n$ . Thus, while transitions such as magnetic quadrupole transitions are small for real photons, they may become very important at large values of  $|\mathbf{q}|$  in electron scattering. Consider electron scattering through  $180^\circ$  where only the transverse electromagnetic multipoles contribute. The cross section is then [3]

$$\frac{d\sigma}{d\Omega} (J_f \leftarrow 0^+) |_{180^\circ} = \frac{\pi\alpha^2}{K_1^2} |\langle J_f || T_J^{\text{mag}}(K_1 + K_2) || 0^+ \rangle|^2 \quad (26)$$

where

$$M_0(q) \equiv \sqrt{4\pi} \int \rho_0(x) j_0(qx) x^2 dx \xrightarrow{q \rightarrow 0} \frac{Z}{\sqrt{4\pi}}.$$

$M_0(q)$  is just the ground state *elastic form factor* of the nucleus. This result was used by Foldy and Walecka [8], and is originally due to Fallieros, Ferrell and Pal [4, 3]. The "semiclassical" model of the spin-isospin resonances, (see Eq. (7)) or the relation  $M_V^2 = M_A^2 = M_P^2$  allows us to extend the result to the axial vector and pseudoscalar matrix elements.

where

$$\begin{aligned} T_{JM}^{\text{el}}(q) &= \frac{1}{q} \int d\mathbf{x} [J_N(\mathbf{x}) \cdot (\nabla \wedge j_J(q\mathbf{x}) Y_{JJ1}^M) + q^2 j_J(q\mathbf{x}) Y_{JJ1}^M \cdot \mu_N(\mathbf{x})] \\ T_{JM}^{\text{mag}}(q) &= \int d\mathbf{x} [\mu_N(\mathbf{x}) \cdot (\nabla \wedge j_J(q\mathbf{x}) Y_{JJ1}^M) + j_J(q\mathbf{x}) Y_{JJ1}^M \cdot J_N(\mathbf{x})]. \end{aligned} \quad (27)$$

$eJ_N(\mathbf{x})$  and  $e\mu_N(\mathbf{x})$  are the nuclear convection current and magnetization densities and  $J \geq 1$  for the transverse multipoles.

Let us first discuss the  $1^-, T = 1$  states. In addition to the contribution of the convection current, \* there is a term

$$[T_{1M}^{\text{el}}(q)]_{\text{spin}} = -\frac{1}{6} \sqrt{\frac{3}{4\pi}} (\lambda_p - \lambda_n) \frac{q}{2Mc} q \sum_{i=1}^A \tau_3(i) [\sigma(i) \odot x(i)]_{1M}. \quad (28)$$

This spin term is usually discarded for photons, however, as it grows with  $q^2$ , it can be made quite large for electrons. (The large isovector magnetic moment  $\lambda_p - \lambda_n = 4.71$  also increases its importance.) This term tends to make the amplitude for electron excitation of the Giant Dipole resonance increase with  $q^2$  while the form factor for the charge part of the operator is a decreasing function of  $q^2$ . The presence of these two competing effects is seen experimentally in both  $\text{O}^{16}$  and  $\text{C}^{12}$  [19, 20, 21]. These experiments tell us there is a very strong component of the matrix elements of  $\sum_{i=1}^A \tau_3(i) [\sigma(i) \odot x(i)]_{1M}$  in the giant resonance region in these nuclei.

The situation is even simpler for the  $T = 1, 2^-$  states. Here the long wavelength form of the isovector part of  $T_{2M}^{\text{mag}}(q)$  is

$$T_{2M}^{\text{mag}}(q)^V \xrightarrow{q \rightarrow 0} \frac{1}{4} \frac{1}{\sqrt{5\pi}} \frac{iq}{2Mc} q \sum_{i=1}^A \tau_3(i) \{[(\lambda_p - \lambda_n)\sigma(i) + \frac{2}{3}l(i)] \odot x(i)\}_{2M}. \quad (29)$$

The cross section to  $2^-$  states thus grows as  $q^4$ . Because of the large isovector moment, one expects the  $T = 1$  transitions to dominate. The magnetic part of the above operator dominates for a similar reason.

\* The convection current contribution can be explicitly evaluated in the Goldhaber-Teller model [3]

$$| \langle 1^- | T_1^{\text{el}}(q) | 0^+ \rangle |^2 = \frac{1}{2} \left( \frac{\hbar^2 q^2}{2\mu} \right) \frac{1}{\hbar\omega} \left( \frac{\omega}{qc} \right)^2 |M_0(q)|^2$$

Recent experiments at Stanford indicate a very rapidly growing well-defined peak of strength (if interpreted as a  $2^-$  state) about  $\frac{1}{2}$  of the sum rule value in the giant resonance region in  $C^{12}$  and  $O^{16}$  [23] \*.

There is therefore some evidence that the supermultiplet of Giant Resonances (or at least some strong remnants of the supermultiplet) may be present in light nuclei. However, several important questions remain unanswered such as, are these resonances systematically present throughout the periodic table? Are there  $T = 1$   $J^\pi = 0^-$  and  $T = 0$ ,  $S = 1$ ,  $J^\pi = 0^-, 1^-, 2^-$  resonances present? What is the effect of strong spin dependences (for example, the strong tensor force component present in most of the more sophisticated nucleon-nucleon potentials) on the Giant Magnetic Resonances? A lot of interesting work remains to be done.

Finally, we note that a Giant Magnetic Quadrupole resonance, exhausting the sum rule of Eq. (3) would help one understand the suppression of low lying  $M2$  transitions discussed recently [23, 24, 25].

The author is very much indebted to Professor L. Foldy for many helpful discussions on this material.

\* A Giant Magnetic Quadrupole state has also been previously predicted by Brown and Vinh-Mau at 19.2 MeV in  $C^{12}$  [17].

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# THE SYMMETRIES OF FORCES AND STATES

ROBERT OPPENHEIMER

*Princeton, New Jersey*

*(Received April 29, 1965)*

The two great new syntheses in our understanding of physics that go back to the earlier years of this century had one important feature in common. They were both characterized by the discovery and eventual understanding of a constant of physics, the velocity of light for special relativity, and the quantum of action for atomic theory, which indicate the limits beyond which Newtonian mechanics on the one hand, and classical physics on the other, no longer apply. In each case these constants play a fundamental part in the formulation of the new laws of physics, and in an understanding of the limitations on physical observation which, in quite different ways, each brings with it.

Today no such discovery, no such fundamental constant, and, of course, no such interpretation characterize the contemporary effort to understand the regularities in the behavior of the particles of physics. One important reason for this state of affairs is that there are only limited situations for which the application of quantum mechanics and relativity lead to well defined predictions. In the case of electrodynamics the predictions are fairly far reaching, and the limits on their credibility as logical deductions fairly well understood. No convincing evidence that there is a minimum interval or maximum mass beyond which these predictions are wrong has yet been found. For the characteristic strong and weak interactions of particle physics, the situation is far more difficult. Despite many and valiant efforts, there are only limited consequences which have been drawn so far from the most general principles of relativity and quantum theory. There has been no indication that these are not true.

Despite valiant efforts, until now more complete predictions based on these general principles have not been possible, nor has it even been possible to prove or disprove that they are consistent with nontrivial physics, or to what extent they define a unique description or a unique

family of descriptions. Models suggested, for instance, by electrodynamics, again despite most valiant efforts that still continue, have not been shown to have a well defined mathematical content which could be compared with experience. None of this is for want of trying.

Both in the discussion of the necessary consequences of the general principles of relativity and quantum theory, and in the efforts to give meaning to some form of Lagrangian or Hamiltonian field theory, the problem of mathematical existence, the problem of defining with adequate rigor the mathematical meaning of the symbolism, has played an ineluctable part, not for the first time in physics, but perhaps for the first time in so essential a way.

Under these circumstances, more circumscribed and more phenomenological approaches have necessarily played a very great part. Two of these go back to discoveries made three decades ago, which have proved fruitful in ordering the observational material, have been increasingly and actively cultivated, and, it appears, may indeed be more closely linked than could have been guessed in their origins. Both are natural outgrowths of the general ideas of relativity and quantum theory.

The first of these suggestions was Yukawa's invention: the meson as the carrier of forces between nucleons. Yukawa based this suggestion on an analogy with electrodynamics. In its qualitative form it is indeed a direct consequence of relativity and quantum theory. For almost two decades, and well after the discovery of some of the relevant mesons, the quantitative elaboration of these ideas was far too limited to perturbation-theoretical methods: the familiar perturbation theory of regarding the emission and absorption of mesons as weak, and the extreme "molecular approximation" of regarding meson mass as very small compared to that of the nucleons, so that the nucleons could be roughly treated as static sources. The discovery of the 3-3 resonance, whose fundamental importance was one of the consequences of the static approximation, stimulated the development of techniques, approximate and in some details arbitrary, which have dominated the discussion of strong interactions ever since. From Chew's static model through the Low equation to the rediscovery by Goldberger and systematic development of dispersion relations, the idea of particle exchange, supplemented by the trivially necessary but nontrivial require-

ment of unitarity, have led to many attempts to understand the existing particles and their properties in terms of the exchange of these particles as the origin of forces that bind them, to the bootstrap, and in many rough treatments to the reciprocal bootstrap. The treatments are rough, for one thing in that they deal with few particles at a time, neglecting many channels which are known to exist, and based on the hope that only the lowest energy phenomena, the nearest lying singularities of the scattering amplitude, will play an important part; for another, they are rough because, having made these assumptions one must either make the dynamics very trivial indeed, or compensate in some not well founded way for the residual effect of the neglected channels and the higher energy phenomena. It is clear that this instrument, though still quite blunt, and not easy even formally to develop into a complete theory, has identified and described important traits of the strongly interacting particles.

The other notion of three decades ago has to do with symmetry. It was based on the detailed analysis of low energy proton-proton scattering by Breit, and was the recognition of the charge independence of nuclear forces; indeed, Cassen and Condon described this independence in a formalism identical to that used for particle spins, in systems of identical particles, with the neglect of spin-orbit coupling. Such a symmetry principle, valid only to the extent that electromagnetic effects could be ignored, is of course reminiscent of the permutation symmetry of quantum theory; but whereas its formal analog, the spin independence of atomic forces, is a recognized limiting case of permutation symmetry, in general charge independence has no known rigorous symmetry for actual charged particles. This was perhaps the first example of an invariance principle categorizing one order of forces which would not apply when weaker electromagnetic forces were considered, and it remained an untroubling puzzle, along with the actual value of the fundamental charge itself. The hierarchy of forces was further enriched by Fermi's theory of beta decay; but it was not until the much later discovery of strangeness, and of parity violation, that the depth, the peculiarity, the at least temporary inexplicability of the hierarchy of forces, and their associated hierarchy of symmetries, were appreciated.

Very soon after the suggestion of mesons, and of charge independ-



ence, Kemmer showed how to combine them in a Lagrangian version of field theory, limiting himself, of course, to the nucleons and the pi mesons, and predicting  $\pi^0$ . At first it was hoped, and the hope died very slowly, that by known methods this Lagrangian would lead to a dynamics. Today it is rather a mnemonic to suggest what particles, what channels, what symmetries, both of space and time and of the internal variables like isotopic spin, should be considered.

With the discovery of strange particles, it very soon became clear that there was a new selection rule and a new quantum number in physics, conserved in strong and electromagnetic interactions. In the following years, almost every low rank semi-simple Lie group found its sponsor and champion; but in the last years there has been little doubt that whatever its mysteries, one such group has its symmetries mirrored in the now known particles of physics: the Gell-Mann-Neeman  $SU(3)$ . Even for the strong interactions this has clearly never been an exact symmetry; important reactions occur which would be forbidden by it; masses are in fact distinct, and for the pseudoscalar mesons superficially vastly distinct, which would be the same if the symmetry were right. These violations cannot be compactly referred to another category of interaction, as can charge independence to electromagnetism; and if they are small as in many applications they seem to be, no hitherto known constant of nature has seemed clearly to define that smallness. Thus there has been a struggle, not unsuccessful, but by no means concluded, to discover what is truly invariant under this group, what is neglected, and how to characterize the forces that are and the forces that are not invariant. One effort, not unnatural in the light of the history of atomic physics, to suggest an answer, has been the notion that the known particles are tightly bound complexes of very much heavier and very much fewer and very much more "fundamental" particles which realize the fundamental representation of the symmetry, and for which the symmetry, though still not perfect, would be a truly good approximation. Insofar as the complete or broken symmetry expresses itself in the properties of known particles, this hypothesis of composition of quarks or aces has been a helpful tool of calculation. Insofar as it represents the hope of reducing the dynamics of the known particles to the properties and forces of more fundamental ingredients, and thus to "compose" the particles as one composes an

atom, or, rather less precisely, even a nucleus, this hope seems doomed, because the conditions under which a model of "composition" can make even approximate sense imply the smallness of the forces, the immutability of the ingredients, and the nonrelativistic features of the bound system. Else we should talk of atoms composed of quanta, and of nuclei composed of neutrinos and electrons.

A second and quite different attempt to learn how to live with the useful, but approximate, and thus somewhat mysterious symmetry has been through the use of model reciprocal bootstraps-model not only because only a few low-lying and often somewhat arbitrarily chosen channels are considered, but because the models have typically been static or quasistatic, and because the input information, though not adequate to establish the symmetry, has borrowed some features, for instance multiplicity, that characterize the symmetry. In these thus abstracted models, one is then asked for self-reinforcing, self-consistent solutions, which means that one seeks characteristic vectors of the crossing matrix for the channels considered, with a characteristic value near or equal to one. In simple cases, and with a variety of input information, suggestive of but not implying a symmetry, one has been able to show that the self-consistency condition implies the symmetry. Often this has been by numerical calculation; but for sufficiently abstract and unrealistic models, and sufficiently simple conditions, the results are algebraic, or can with a little numerical encouragement be further simplified to algebraic form. So one understands how  $SU(2)$  may be generalized to  $SU(3)$ ; so one understands octet enhancement in broken symmetries, and in determining the rough relations between reaction rates in strong, electromagnetic and weak interactions, and the pattern of mass relations within  $SU(3)$  multiplets, and within isotopic multiplets. It would be too simple to say that these models, which neglect so much, identify what breaks the symmetry; but one thing neglected in them is the recoil of the baryons, and thus the ratio of meson mass to baryon mass. As in the static approximation of long ago, this is the sort of smallish but not small number which seems to be needed.

All of this has been made far more frantic, and far more susceptible of misinterpretation, by the discovery of  $SU(6)$  symmetry last summer. Attempts had, of course, long been made to derive the internal sym-

metries from the Poincaré group, typically from its discrete elements – of course, unsuccessful attempts. More recently, attempts have been made to marry the exact symmetry of the Poincaré group with broken internal symmetries. There are now adequate mathematical proofs that such structures are either trivial and that there is no marriage, or involve physical consequences wholly alien to our experience with the physics of particles; yet these efforts, largely by four brilliant Turks, may yet, as a half millennium ago, in their turn lead to the discovery of America. For  $SU(6)$  has not been a useless group, any more than  $SU(3)$  before it; and there are many examples in older parts of physics where symmetries which cannot be found by staring at the Lagrangian of the theory, which does in those cases exist, still give a useful characterization of the states to which this Lagrangian leads. It is not an uncommon experience in quantum mechanics, when a rough symmetry can be discerned in a poor approximation to the actual forces, that the states of the system either ignore or very fully realize this rough symmetry. Striking examples are in the properties of quantum liquids, where the quantum properties and order are not visible above the transition temperature, and wholly dominate what happens below it. Historically the closest, and historically the most relevant theory is Wigner's theory of super multiplets and  $SU(4)$ . Indeed, it was the rather amazing success in characterizing nuclear states, even when spin-orbit and electromagnetic interactions would appear to be not at all negligible, that led to the consideration of  $SU(6)$  by Gürsey and Radicati.

Here, where one is dealing with particle physics, often with reasonably high momenta, and with particle creation and annihilation, the prospects for successfully stealing the spin, which is a part of some of the generators of the Poincaré group, without getting into trouble, seem even more remote; but  $SU(6)$  has been a suggestive symmetry, and its far more numerous and restricting regularities are not really much less reliable than those of  $SU(3)$ . There have been many efforts, despite the general theorems, to enlarge  $SU(6)$  to a group that should contain it and the Poincaré group. These all lead to trouble, as they must; and it is an open question what role they can play in heuristic suggestions of interrelations going beyond  $SU(6)$  that are reflected in reality. This is not the occasion to review the successes in predictions of mass splittings, electromagnetic and weak couplings, and strong



reaction rates, or of the places where there are most bothersome discrepancies. What is clear is that we are here, as in  $SU(4)$ , and in the models of quantum liquids, not dealing with an abstract symmetry of a Lagrangian, but with rough symmetries, often not nearly as rough as as we might guess, of certain of the states. In a view which makes the exchange of particles responsible for their existence, the symmetry of states, and the symmetry of forces, are not unrelated. Thus here again, though again in purely numerical form, Dashen and Frautschi have been able to extract many  $SU(6)$  results from a reciprocal bootstrap containing only the barest  $SU(6)$  ingredients, and not itself  $SU(6)$  invariant, and to show, as B. W. Lee has done more algebraically, that the weak and electromagnetic currents of the particles satisfy the algebraic relations of the generators of the compact group  $U(12)$ .

This leaves the question of symmetries largely open for the future. It is, for instance, not clear that for high energy phenomena, unless they are completely dominated by a few low lying states in crossed channels, the symmetries will emerge more fully; nor is it clear in what measure the patterns of weak interactions are fully encompassed in the symmetries of the particles. Thus the discovery last summer of weak effects in the 2 pion decay of the long-lived K meson, indicating that combined parity CP is not conserved, and suggesting time reversal noninvariant forces, is not definitively understood. These effects may derive, as suggested by many, from small corrections to the weak forces themselves, or from time noninvariant electromagnetic interactions, or, in my opinion far more hopefully, as suggested by T. D. Lee, in a lack of correspondence between the conjugation operators connected with baryon number, perhaps lepton number, and electric charge, the three rigorously conserved quantities characteristic of particle physics.

When we think how far we are in this search for order, how still farther from any clear notion of what, beyond or against quantum theory and relativity, we should be discovering, and of how mysterious the hierarchies of interactions and symmetries still seem, we know that a unitary and nonarbitrary description of the phenomena of particle physics is still a great work for the physicists. We may remember Whitehead, gratefully "to leave the vast darkness unobscured", but not for long, and surely not forever.



The views expressed in this note will surely not be wholly, perhaps not even widely shared by Weisskopf, for whom it is written. I know that he will share the hope, on my part both earnest and confident, for the welcome part he will play in the great work ahead.

# THE GROUP $S_3$ AND STRONG INTERACTIONS

YOSHIO YAMAGUCHI

*Institute for Nuclear Study, University of Tokyo  
Tanashi-machi, Kitatama-gun, Tokyo, Japan*

*(Received April 29, 1965)*

## 1. ISO-SPIN

The concept of iso-spin, which is one of the foundation stones of modern particle physics, was introduced by W. Heisenberg [1] in 1932 immediately after the discovery of the neutron. However, curious enough, the notion of charge independence has never been fully appreciated in *nuclear physics* for surprisingly long years since then. It would be important to remark that the equality of p-p, p-n (and n-n) forces at least in  $^1S$ -state is known (experimentally) at the later half of 1930s [2]. Of course one can quote several works in which the charge independence was rightly emphasized and utilized, among which (a) the symmetrical meson theory by Kemmer [3] and (b) the super-multiplet scheme by Wigner [4] are most important. (One may add further the charge independent treatment of meson-nucleon scattering by Heitler [5].) It should be needless to point out that (a) was the prototype of charge independent theory of (isovector) meson-(iso-spinor) nucleon interactions. Whereas importance of (b) has been eclipsed (till unnecessary degree) by Bohr's picture of atomic nuclei (pre- and mid-war periods) and by the great success of the shell-model (post-war period). Only at relatively later times, significance of (b) began to be realized not only in nuclear physics but also in particle physics [6].

Serious use of the charge independence in particle physics was introduced by Fermi in his celebrated analysis of the pion-nucleon scattering experiment [7]. In nuclear physics one must refer to the important contribution by Adair [8]. Since then, Nakano, Nishijima and Gell-Mann proposed a charge independent theory of strong interactions with an important new quantum number, strangeness. What followed then is so well-known that one need not so say further.

Charge independent theories are invariant under all rotations in the iso-spin space; i.e., they satisfy  $O_3$ , or equivalently  $SU_2$ -invariance. Nowadays it is fashionable to talk about higher symmetries in particle physics. It is interesting to notice the following chronological intervals:

- 1932 symmetry under  $SU_2$  (iso-spin)
- 1959 (approximate) symmetry under  $SU_3$  (unitary symmetry)
- 1964 (approximate) symmetry under  $SU_6$ ,  $SU_9$ ,  $SU_{12}, \dots$

So many years were necessary to appreciate  $SU_2$ -symmetry, while such a rush to climb up higher and higher symmetries in these days!

We usually assume the invariance under *all* rotations in the charge space (three dimensional iso-spin space) in charge independent theories. Is it really necessary to require the invariance for all rotations? Or in other words: Why is it not sufficient to ask for invariance for some limited number of rotations? This question was in fact raised by Case, Karplus and Yang [9] (abbreviated hereafter as CKY) just after the proposal of the strangeness scheme of strongly interacting particles. These authors discussed the symmetry property in the particle world in terms of finite groups. Along such a line of argument we shall develop our discussion in this paper. (Another aspect of iso-spins – a problem of locally dependent charge-axis or of “iso-spin-gauge” – was discussed by Yang and Mills [10]. We shall, however, not concern with it in the present paper.)

## 2. USE OF FINITE GROUPS

The use of finite groups is not at all new in physics. Finite groups were beautifully applied to crystal structure, and atomic as well as molecular physics. Finally but most importantly, the symmetry group  $S_n$  of order  $n$  is indispensable for the quantum mechanical treatment of the system of  $n$  identical constituents (fermions or bosons).

As noticed in § 1, CKY treated the internal symmetry of the particle world on the basis of certain finite groups. They noticed an important theorem as follows.

Suppose that “multiplets” of (elementary) particles belong to irreducible representations of a finite group, say, tetrahedron group  $T$ , and strong interactions are invariant under this group. A triplet pion ( $\pi^+ \pi^0 \pi^-$ ) corresponds to three dimensional irreducible representation

3 of T, where electric charge is used to distinguish (in an ad hoc manner) three independent states of the pion. Consider the elastic pion-pion scattering. The two pion states can be classified according to the irreducible representations of T:

$$3 \times 3 = 1 + 3_a + 3_s + 2 \quad (\text{for T}). \quad (2.1)$$

Each irreducible set is given by

$$1 \quad (\pi^+ \pi^- - \pi^0 \pi^0 + \pi^- \pi^+)/\sqrt{3} \quad (2.2)$$

$$3_a \quad \begin{cases} (\pi^+ \pi^0 - \pi^0 \pi^+)/\sqrt{2} \\ (\pi^+ \pi^- - \pi^- \pi^+)/\sqrt{2} \\ (\pi^0 \pi^- - \pi^- \pi^0)/\sqrt{2} \end{cases} \quad (2.3)$$

$$3_s \quad \begin{cases} (\pi^+ \pi^0 + \pi^0 \pi^+)/\sqrt{2} \\ (\pi^+ \pi^- + \pi^- \pi^+)/\sqrt{2} \\ (\pi^0 \pi^- + \pi^- \pi^0)/\sqrt{2} \end{cases} \quad (2.4)$$

$$2 \quad \begin{cases} (\pi^0 \pi^0 + \varepsilon \pi^+ \pi^+ + \varepsilon^2 \pi^- \pi^-)/\sqrt{3} \\ (\pi^0 \pi^0 + \varepsilon^2 \pi^+ \pi^+ + \varepsilon \pi^- \pi^-)/\sqrt{3} \end{cases} \quad (2.5)$$

where

$$\varepsilon = \sqrt[3]{1} \equiv \frac{1}{2}(-1 + \sqrt{3}i).$$

It is interesting to compare the result (2.1) with that of the usual iso-spin formalism:

$$\begin{array}{ccccccc} 3 \times 3 = & 1 & + & 3 & + & 5 & (\text{for } SU_2) \\ & \uparrow & & \uparrow & & \uparrow & \\ & I = 0 & & I = 1 & & I = 2 & (\text{total iso-spin}). \end{array}$$

It is clear that **1** or **3<sub>a</sub>** for T is identical with 1 or 3 for the conventional charge independent case, respectively, and **3<sub>s</sub>** and **2** for T are linear combinations (which are not always diagonal with respect to electric charge) of five independent states with  $I = 2$ :

$$\pi^+ \pi^+, (\pi^+ \pi^0 + \pi^0 \pi^+)/\sqrt{2}, (\pi^+ \pi^- + 2\pi^0 \pi^0 + \pi^- \pi^+)/\sqrt{6}, \\ (\pi^0 \pi^- + \pi^- \pi^0)/\sqrt{2}, \pi^- \pi^-.$$

The scattering matrix for pion-pion scattering must have the following form,



	1	$3_a$			$3_s$			2	
1	$S_1$	0	0	0	0	0	0	0	0
	0	$S_3$	0	0	0	0	0	0	0
$3_a$	0	0	$S_3$	0	0	0	0	0	0
	0	0	0	$S_3$	0	0	0	0	0
	0	0	0	0	$S_5$	0	0	0	0
$3_s$	0	0	0	0	0	$S_5$	0	0	0
	0	0	0	0	0	0	$S_5$	0	0
	0	0	0	0	0	0	0	$S'_5$	0
2	0	0	0	0	0	0	0	0	$S'_5$

(2.6)

where the interactions are, as assumed from the beginning, invariant under the group T. However (2.6) violates in general the conservation law of electric charge. Hence, one postulates further that the electric charge should be conserved in any processes and finds

$$S'_5 = S_5. \quad (2.7)$$

The scattering matrix (2.6) with (2.7) has precisely the same form as should be derived from the conventional charge independent description of the pion-pion scattering.

Summarizing above discussion, one can reach the theorem: Suppose a theory of strong interactions invariant under the group T is given, and the conservation of electric charge (which should be *appropriately* assigned to each member of multiplets of particles) is required. Then the original theory becomes charge independent, i.e., it is invariant not only under T (finite group of the order 12) but also under  $SU_2$  (Lie group of rank 2). A complete proof of this theorem as well as precise conditions on which the theorem holds will be left for the reader.

Analogous situations can be found in any finite group: If multiplets of particles are assigned to irreducible representations of a finite group, under which a theory is invariant, the invariance requirement for this group does not in general guarantee the conservation law of electric charge. An additional postulate of the charge conservation law makes an original symmetry to enlarge to a higher symmetry (in the above

example: from the tetrahedron symmetry to  $SU_2$ -symmetry) with a corresponding enlargement of the irreducible representations. In this way the charge independence (charge-symmetry or unitary symmetry, etc., depending on the choice of a finite group) is restored [9, 11].

In the next paragraph, we shall fully use this sort of situation. CKY imposed in their paper the condition: the pion triplet should be irreducible member of finite groups. Hence the smallest group which CKY discussed was the tetrahedron group. Whereas we shall not accept this limitation, and can try to consider other finite groups whose orders are smaller than the order of tetrahedron group.

### 3. $S_3$ AND LEVEL-SCHEME

It is interesting to remark that finite groups contain only limited number of operations, so that invariance requirement under a certain finite group is less restrictive than that under a (corresponding) Lie group, and hence the former can be much richer than the latter. This situation would easily be visualized by comparing, e.g., figures on a plane with  $C_n$ -symmetry (invariant under rotation of  $2\pi/n$ ,  $n$  being an integer) and  $C_\infty$ -symmetry (rotationally symmetric figures) around the axis perpendicular to the plane.

We propose to discuss internal symmetry properties in the particle world based on a finite group, although symmetries are usually treated by a Lie-group (or more appropriately by Lie-algebra). Any finite group is always equivalent to either  $S_n$  (the symmetry groups of order  $n$ ) or its subgroup. We also appeal to the principle of simplicity. Hence we may discuss  $S_n$  with  $n = 1, 2, 3, \dots$  till we reach a sensible model of particle physics with smallest integer  $n$ .  $S_1$  or  $S_2$  is too simple to contain irreducible representations with more than one dimension. Next we must try  $S_3$ , which is rich enough as will be seen in § 4.

Before discussing our theory with  $S_3$ -invariance, we shall consider a simple quantum mechanical problem which serves to clarify the situation: to realize a "broken-symmetric" particle-world upon the basis of "symmetric" dynamics.

Consider a hypothetical molecule consisting of three identical atoms, and the pattern of its Raman spectra. This molecule has  $3 \times 3 = 9$  degrees of freedom. 3 degrees of freedom are attributed to the center of mass motion, hence have nothing to do with the internal

excitation of the molecule. Other 3 degrees of freedom may be separated to describe (rigid-) rotations of the molecule as a whole, which are approximately independent on the remaining three ( $= 9 - 3 - 3$ ) internal degrees of freedom at sufficiently low excitation. Thus, we only have three degrees of freedom relevant for internal excitation of our molecule. We introduce three canonical variables  $x, y, z$  to describe infinitesimal deviations of three atoms from their equilibrium positions. We can choose  $x, y, z$  to be completely symmetric, since the molecule consists of three identical atoms. In other words, the Hamiltonian  $H(x, y, z)$  satisfies

$$H(x, y, z) = H(x, z, y) = H(y, x, z) = H(z, x, y) = H(y, z, x). \quad (3.1)$$

Namely,  $H$  is invariant under any permutations among  $x, y, z$ , which form the group  $S_3$  with order  $3! = 6$ .

$S_3$  has 3 different regular irreducible representations:  $1_s$ ,  $1_a$ , and 2 ( $6 = 1^2 + 1^2 + 2^2$ ). It is clear that the symmetrical coordinate  $(x, y, z)$  introduced above is a reducible representation of  $S_3$ . A decomposition into irreducible representations is obtained e.g. by the linear combination

$$\left. \begin{aligned} x_A &= \frac{1}{\sqrt{3}}(x + y + z) \\ x_n &= \frac{1}{\sqrt{6}}(2x - y - z) \\ x_p &= \frac{1}{\sqrt{2}}(-y + z) \end{aligned} \right\} \quad (3.2)$$

It is easy to prove that  $x_A$  is the identical representation while  $(x_n, x_p)$  spans 2 dimensional irreducible representation of  $S_3$ . In terms of these new variables the basic Hamiltonian  $H(x_A; x_n, x_p)$  is symmetric for interchange between  $x_n$  and  $x_p$ :

$$H(x_A; x_n, x_p) = H(x_A; x_p, x_n), \quad (3.3)$$

but – very important to remark – not in general symmetric between  $x_A$  and  $x_n$  or  $x_p$ :

$$H(x_A; x_n, x_p) \neq H(x_n; x_A, x_p). \quad (3.4)$$

Now the pattern of Raman spectra of the molecule is evident: lowest three Raman frequencies consist of a singlet and a doublet corresponding to the basic symmetry  $1+2$  (1 for  $x_A$ , 2 for  $x_n, x_p$ ).

This molecular problem gives a natural way to derive the "asymmetric" lowest excited states (singlet+doublet rather than triplet) from the Hamiltonian  $H(x, y, z)$  with complete symmetry among  $(x, y, z)$  (see (3.1)). In the subsequent section, we build up a model of strong interactions analogous to the tri-atomic molecule described here.

#### 4. MODEL OF STRONG INTERACTIONS

We construct a model of strong interactions [12, 13] upon  $S_3$ -symmetry appealing to a molecular analogue given in § 3. For this purpose we shall introduce three basic Dirac fields  $\psi_1, \psi_2, \psi_3$ . We do not admit here any a priori distinction among these basic fields – cf. the situation of symmetrical coordinates  $(x, y, z)$  in § 3. In other words, our theory must be invariant under all (6) permutations among the fields  $\psi_1, \psi_2, \psi_3$ , i.e., invariant under  $S_3$ . Therefore, the name of chaos-fields may be appropriate for  $\psi_1, \psi_2, \psi_3$ .

We postulate that each of three chaos-fields carries baryon number  $N = 1$  (or  $1/3$  as in the case of quarks or aces [14] if one wishes), and the theory should be  $N$ -conserving.

As was the molecular case in § 3, chaos-fields are reducible with respect to  $S_3$ . Irreducible bases, which we shall call the Sakata-fields, can easily be formed by

$$\left. \begin{aligned} \psi_A &= \frac{1}{\sqrt{3}} (\psi_1 + \psi_2 + \psi_3), \\ \psi_n &= \frac{1}{\sqrt{6}} (2\psi_1 - \psi_2 - \psi_3), \\ \psi_p &= \frac{1}{\sqrt{2}} (-\psi_2 + \psi_3). \end{aligned} \right\} \quad (4.1)$$

$\psi_A$  is the identical representation and  $(\psi_n, \psi_p)$  is the two-dimensional irreducible representation of  $S_3$ . Anti-fields  $\bar{\psi}_A, \bar{\psi}_n, \bar{\psi}_p$  (are assumed to) behave as  $\psi_A, \psi_n, \psi_p$  under permutations.

Next, we discuss a two-body system, i.e., a reduction of product



representation  $(1+2) \times (1+2)$  to write down Fermi-type interactions between Sakata-fields. We find following irreducible sets for a two-body system of Sakata- and anti-Sakata particles:  
one dimensional representations

$$\left. \begin{aligned} &\bar{\psi}_A \psi_A, \\ &\frac{1}{\sqrt{2}} (\bar{\psi}_p \psi_p + \bar{\psi}_n \psi_n), \\ &\frac{1}{\sqrt{2}} (\bar{\psi}_p \psi_n - \bar{\psi}_n \psi_p); \end{aligned} \right\} \quad (4.2)$$

two dimensional representations

$$\left\{ \begin{array}{l} \bar{\psi}_A \psi_n \\ \bar{\psi}_A \psi_p \end{array} \right\} \quad \left\{ \begin{array}{l} \bar{\psi}_n \psi_A \\ \bar{\psi}_p \psi_A \end{array} \right\} \quad \left\{ \begin{array}{l} \frac{1}{\sqrt{2}} (\bar{\psi}_p \psi_p - \bar{\psi}_n \psi_n) \\ \frac{1}{\sqrt{2}} (\bar{\psi}_p \psi_n + \bar{\psi}_n \psi_p). \end{array} \right\}$$

A  $S_3$ -invariant Hamiltonian of Fermi-type is in general given by

$$\begin{aligned} H = & a(\bar{\psi}_A \psi_A)^2 + b(\bar{\psi}_A \psi_A)(\bar{\psi}_p \psi_p + \bar{\psi}_n \psi_n) \\ & + \{b'(\bar{\psi}_A \psi_A)(\bar{\psi}_p \psi_n - \bar{\psi}_n \psi_p) + \text{h.c.}\} \\ & + c(\bar{\psi}_p \psi_p + \bar{\psi}_n \psi_n)^2 \\ & + c'(\bar{\psi}_p \psi_p + \bar{\psi}_n \psi_n)(\bar{\psi}_p \psi_n - \bar{\psi}_n \psi_p) \\ & + d(\bar{\psi}_p \psi_n - \bar{\psi}_n \psi_p)^2 \\ & + [f'\{(\bar{\psi}_A \psi_p)^2 + (\bar{\psi}_A \psi_n)^2\} + \text{h.c.}] \\ & + g\{(\bar{\psi}_p \psi_p - \bar{\psi}_n \psi_n)^2 + (\bar{\psi}_p \psi_n + \bar{\psi}_n \psi_p)^2\} \\ & + [h\{(\bar{\psi}_A \psi_n)(\bar{\psi}_p \psi_p - \bar{\psi}_n \psi_n) + (\bar{\psi}_A \psi_p)(\bar{\psi}_p \psi_n + \bar{\psi}_n \psi_p)\} + \text{h.c.}], \end{aligned} \quad (4.3)$$

where the space-time operators sandwiched between  $\bar{\psi}$  and  $\psi$  are omitted for brevity, but the baryon number conservation (or in brief,  $N$ -conservation) is properly taken into account.

At this stage we shall make an (a priori) assignment of "p"-number  $N_p$ :  $\psi_p$ -field has  $N_p = 1$  while  $\psi_A$  and  $\psi_n$  have  $N_p = 0$ . We impose the condition that sum of the "p"-number should be conserved in all reactions. Then we have to set the following conditions in the inter-

action (4.3):

$$\left. \begin{aligned} b' = c' = f' = h = 0 \\ d = -g \end{aligned} \right\} \quad (4.4)$$

and obtain the interaction of the form

$$\begin{aligned} H = & a(\bar{\psi}_A \psi_A)^2 + b(\bar{\psi}_A \psi_A)(\bar{\psi}_p \psi_p + \bar{\psi}_n \psi_n) \\ & + c(\bar{\psi}_p \psi_p + \bar{\psi}_n \psi_n)^2 \\ & + f\{(\bar{\psi}_A \psi_p)(\bar{\psi}_p \psi_A) + (\bar{\psi}_A \psi_n)(\bar{\psi}_n \psi_A)\} \\ & + g\{(\bar{\psi}_p \psi_p - \bar{\psi}_n \psi_n)^2 + 2(\bar{\psi}_p \psi_n)(\bar{\psi}_n \psi_p) + 2(\bar{\psi}_n \psi_p)(\bar{\psi}_p \psi_n)\}. \end{aligned} \quad (4.5)$$

To derive the result (4.5), it is clear that the “n”-number ( $N_n$ ) conservation can be used equally well:  $\psi_n$ -field has  $N_n = 1$  while  $\psi_p$  or  $\psi_A$  has  $N_n = 0$ . This demonstrates the symmetry between  $\psi_p$  and  $\psi_n$ . As a matter of fact,  $N_p$  is, in practice, electric charge  $Q$ . Nevertheless we do not like to set  $N_p = Q$  simply because we are discussing the particle-world in which only strong interactions exist while the electromagnetism is not yet introduced. Moreover, any orthonormal linear combinations of  $\psi_p$  and  $\psi_n$  are equally good for two dimensional representation of  $S_3$ . In this sense the electric charge  $Q$  cannot be fixed by strong interactions only. Similar situation exists in the conventional charge independent theories in which only strong interactions are considered: the third axis (or charge axis) of the iso-spin space is quite arbitrary for strong interactions as it should be, and the introduction of electromagnetic couplings fixes the third axis.

Let us observe closely the resulting interaction (4.5). For example, we can immediately identify  $\psi_A$ ,  $\psi_n$ ,  $\psi_p$  to be basic fields in the old Sakata model: we can attribute to basic Dirac particles described by  $\psi_A$ ,  $\psi_n$ ,  $\psi_p$  the same quantum numbers (iso-spin and strangeness) as the physical  $\Lambda$ -particle and physical nucleon, and all observed baryons, mesons and their excited states will be composed of these basic fields and their anti-fields. It is important to notice that (4.5) is charge independent ( $SU_2$ -invariant) and strangeness-conserving. Namely,  $S_3$ -invariance and  $N_p$ - (or  $N_n$ -) conservation lead to a  $SU_2$ -invariant and strangeness conserving theory provided that the basic  $S_3$ -invariant interactions are of  $N$ -conserving and of Fermi-type. This theorem is analogous to that for tetrahedron group described in § 2. It would be

strongly emphasized that iso-spins and strangeness are a posteriori introduced into our theory (at the later stage of (4.5) rather than at (4.3)).

A modified version of the above theorem may be worth to mention: Suppose that the basic interactions among basic Sakata-fields are  $S_3$ -invariant,  $N$ -conserving and  $N_A$ -conserving ( $N_A$  being “ $A$ ”-number defined as:  $\psi_A$  has  $N_A = 1$  while  $\psi_n$  or  $\psi_p$  has  $N_A = 0$ ).  $-N_A$  is equal to strangeness and  $N_A$ -conservation is equal to strangeness-conservation. Then interactions are  $SU_2$ -invariant and  $\psi_A$  can be regarded as iso-scalar while  $(\psi_n, \psi_p)$  as iso-doublet in the conventional iso-spin formalism. Notice in this version the condition that interactions should be of Fermi-type can be eliminated.

Another interesting way of using this  $S_3$ -invariant model is to identify  $\psi_A, \psi_n, \psi_p$  with so-called quarks [14]. Accordingly,  $\psi_A, \psi_n, \psi_p$  should have  $N$ -number equal to  $\frac{1}{3}$  and electric charges  $Q$  which are multiples of  $\frac{1}{3}$  (but iso-spin and strangeness are not yet introduced).  $S_3$ -invariance as well as  $N$ - and  $Q$ -conservation reduces a general interaction (4.3) to charge independent and strangeness-conserving one (4.5). From quark-fields  $\psi_A, \psi_n, \psi_p$  one can construct the octet version of *broken*  $SU_3$ -symmetry.

It should be emphasized that basic Sakata-fields  $\psi_A, \psi_n, \psi_p$  are not unitary symmetric but already broken into singlet plus doublet (broken  $SU_3$ ) in these models just mentioned. However, it is clear that if the chaos-fields  $\psi_1, \psi_2, \psi_3$  are used to describe the Hamiltonian the particle-world looks perfectly symmetric.

We have thus succeeded to explain how to derive a broken symmetric particle-world from the symmetric basis (“chaos”). Here we must admit that our model contains no a priori reasons why deviation from the unitary symmetry is so slight. This fact must, unfortunately, be incorporated with a model on the empirical ground.

Above discussions are based on the fact that a Lie group is, naively speaking, split into a finite group and Abelian group(s) (corresponding to conservation law(s)) (e.g.,  $SU_2 \leftrightarrow S_3$  and  $Q$ -conservation).

Finally we remark on the full use of all possible regular irreducible representations of  $S_3$ :  $1_s, 1_a, 2$ . Provided that the group  $S_3$  is of fundamental significance to particle physics, we may postulate the existence of four basic (Sakata-) particles (for strongly interacting

particles) corresponding to three inequivalent irreducible representations of  $S_3$ . These four basic particles exhibit the pattern

$$1 + \underbrace{\{1 + \underbrace{2}_{SU_2}\}}_{\text{weakly broken } SU_3} \quad (4.6)$$

$$\underbrace{\hspace{10em}}_{\text{badly broken } SU_4}$$

and form the basis of "broken"  $SU_4$ -scheme. It is interesting to notice that this pattern (4.6) fit four leptons (muon, electron and two different neutrinos) as well. It is evident that four basic particles for strongly interacting particles and four leptons show remarkable parallelism, which may be regarded as a new form of the baryon-lepton symmetry.

It is my great pleasure to dedicate the present article to Professor V. F. Weisskopf who made really unsurpassed contribution to CERN as its director general and from whom I learnt a great deal of physics and others. This paper was based on my talk given at Trieste (I.C.T.P.), Napoli and Orsay, etc., where many useful comments were given to me. I would like to express my gratitude to Prof. A. Salam, Prof. J. Prentki, Prof. B. Vitale, and Prof. M. Jean for their hospitality.

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# DIFFRACTION MODELS FOR DIRECT NUCLEAR AND HIGH ENERGY PROCESSES

ERNEST M. HENLEY

*Department of Physics, University of Washington, Seattle, Washington*

*(Received April 30, 1965)*

## 1. INTRODUCTION

In the past few years the analytic properties of the  $S$  matrix have been explored in detail, especially in applications to strong interactions at high energy. It is not these formal relationships which concern us here, but rather some simple models which it is profitable to consider for their "Anschaulichkeit". In particular we shall examine relations between various processes which are able to give one insight into reaction mechanisms and may yield quantitative information for the partial and total cross sections without having many (if any) adjustable parameters.

The characteristic of direct scatterings, including inelastic scattering and reactions, is that they generally proceed in a time of the order of that taken by the incident particle to cross the strong interaction region [1]. Typically, such processes are characterized by small momentum transfers, so that the differential cross sections tends to be peaked in the forward direction. At high energies these scatterings are often analyzed by the exchange of the lowest possible mass particle and are called "peripheral" reactions. At lower energies their analysis has most often been carried out with distorted waves and the processes are often called surface reactions. The reasons for these names are that the small momentum transfer implies grazing collisions and the many open competing channels mean that any particle which penetrates deeply into the strong interaction region has a small probability of causing direct reactions. We would like to stress that the analyses of low and high energy direct reactions are closely related, especially if competing channels are taken into account in peripheral processes.

The analyses of the direct reactions we study do make use of distorted waves to describe the strong interactions which occur in the initial and final states. However, unlike full DWBA treatments, which depend on a large number of unknown parameters [2], the description is characterized by few, if any, adjustable constants. A diffraction approach is used, which is especially applicable for small wavelengths, large absorption by competing channels, and a short range transition potential or operator [3, 4].

The relationships we shall discuss apply both at medium and high energies and allow one to draw on the elastic scattering, for instance, to describe inelastic events and reactions. Furthermore, when more than two particles occur in the final state, similarities to a reaction with two particles can sometimes be used to advantage. Examples of these cases will be given below.

## 2. CONFIGURATION SPACE DIFFRACTION MODEL

A diffraction model for inelastic scattering can be derived from a distorted wave or adiabatic approximation [5, 6]. We would like to adopt a more heuristic approach first, which is based on a physical

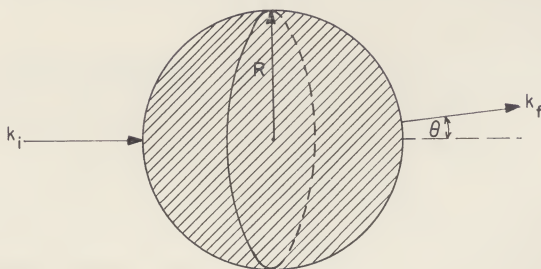


Fig. 1. Region of strong absorption. This is taken to be a sphere of radius  $R$ . The incident, and final relative momenta are also shown, together with the scattering angle  $\theta$ .

model. The usefulness of this development is its simplicity, although it cannot replace a more formal treatment if one wants to place the model on a firm base and understand its shortcomings.

In its basic form, the diffraction model assumes that the configuration region responsible for the process can be fairly well localized.

This occurs typically because several conditions are simultaneously satisfied: 1) the absorption is large, 2) the region of strong absorption can be localized and is bounded by a fairly sharp surface (see Fig. 1), 3) the transition under study is due to a short range interaction so that the relevant form factor (e.g. due to the particle exchanged) falls off quite fast outside this surface. If the above conditions are satisfied for incident and outgoing particles of short wavelength, then the spatial contributions to the matrix element will have a maximum close to the boundary of the absorbing region, and for small scattering angles,  $\theta$ , the Fraunhofer approximation obtains [6]. If the  $z$ -axis is taken along the direction of the incident beam, then for small scattering angles the large absorption tells us that the dominant contribution will come from a surface region close to the great circle lying in the  $x$ - $y$  plane and shown in Fig. 1. If, for simplicity, we assume spinless particles in the incident and outgoing channels then the sharp absorption region suggests that we may use plane waves outside this region and its shadow [4]. The cross section is thus proportional to ( $\hbar = c = 1$ )

$$I(\theta) = \sum_m \left| \int d^3r' e^{-ik_r \cdot r'} \phi_j^m(r') \delta(r' - R) \delta(\theta' - \tfrac{1}{2}\pi) \right|^2 \quad (1a)$$

$$\propto \sum_m \left| \int_0^{2\pi} e^{-ikR \sin \theta \cos \varphi} \phi_j^m(R, \tfrac{1}{2}\pi, \varphi) d\varphi \right|^2$$

$$\propto \sum_m |J_m(kR \sin \theta) Y_j^m(\tfrac{1}{2}\pi, 0)|^2, \quad (1b)$$

where  $\phi_j^m$  describes the appropriate propagator or wavefunction of the transferred particle (e.g. meson or neutron) in configuration space. If the transfer angular momentum is characterized by  $j$  then  $\phi_j^m$  is generally proportional to the spherical harmonic  $Y_j^m$ . The integral in Eq. (1a) is over the great circle of Fig. 1 and gives the Bessel function of order  $m$ . Because  $\mathbf{k}_i \cdot \mathbf{R}$  is zero along the great circle in the  $x$ - $y$  plane, the momentum  $k$  should be characteristic of the momentum  $k_f$  in the final state. However, for high incident momenta and small energy losses  $k_i \approx k_f \approx k$  and this assumption has been made in Eq. (1). The above derivation is clearly fairly crude and makes no direct use of elastic scattering parameters other than the strong absorption. It is for this reason that the model serves as a useful guide

for the angular distribution to be expected in inelastic scattering, as well as in nuclear and high energy reactions. Furthermore, the model can be generalized to large angles and to cases for which  $\mathbf{k}_f$  is not close to  $\mathbf{k}_i$  by various simple devices [4] such as using  $\mathbf{k}_i + \mathbf{k}_f$  as the  $z$ -axis, as suggested by Glauber [7].

For medium energy nuclear reactions, such as  $(\text{He}^3, n)$  and  $(\alpha, d)$ , the above formula or slight variations of it have been obtained and used by many authors [8]. For inelastic scattering of alpha particles, Eq. (1) was obtained by Blair [6] in the adiabatic approximation. However, his approximation is on the scattering amplitude rather than on the wavefunction, and allows him to find absolute differential cross sections. This is also possible in our model, if instead of using a ring, radial integrals are carried out in Eq. (1a) outside the sharp absorption region and its shadow [4]. However, in a realistic physical scattering the transition surface is not sharp. Although the use of plane waves outside the absorbing region can be justified on a W.K.B. approximation, there may be appreciable contributions to the radial integral from the transition region, where plane waves are no longer appropriate. It is for this reason that the model is less trustworthy for evaluating the magnitude of cross sections than it is for angular distributions.

Despite its shortcomings, the simple diffraction model outlined above suggests several features which are found experimentally, but which are not easily deduced from a full DWBA treatment. An example is the so-called Blair phase rule [6]. Since  $Y_j^m(\frac{1}{2}\pi, 0)$  vanishes unless  $i+m$  is even, the angular distribution in the region  $kR \sin \theta > m$  (asymptotic region) for even  $j$  is predicted to be out of phase with the elastic (diffraction) and differential cross section for odd  $j$  transitions. This occurs because in the asymptotic region  $J_m(x)$  is proportional to  $\cos x$  for  $m$  even and to  $\sin x$  for  $m$  odd. In addition, if radial integrals are performed one finds that nuclear reactions with a transfer of angular momentum  $j = |\mathbf{k}_f - \mathbf{k}_i|R$  are preferred [4] if no other selection rules operate, and that natural odd parity states are suppressed in the forward direction [9].

### 3. DIFFRACTION MODEL IN ANGULAR MOMENTUM SPACE

Although generalizations for a smooth absorbing edge [5, 10] and for second order effects in inelastic scattering [5] are also possible,



these considerations, as well as those due to spins in the initial and final channels, are more easily carried out with an angular momentum decomposition. This has the added advantage that it allows a direct comparison with, or use of elastic scattering parameters. No assumption about the absorbing region is necessary. However, the equivalent statement of a dominant contribution from a radius  $R$  is the existence of a critical angular momentum  $L \approx kR - \frac{1}{2}$ , from which the major contribution arises.

The representation we refer to has been developed by several authors for nuclear physics applications [5, 11] and for high energy work [3, 12]. Although differences between these approaches do exist, they are basically similar and it is this similarity which we wish to explore here.

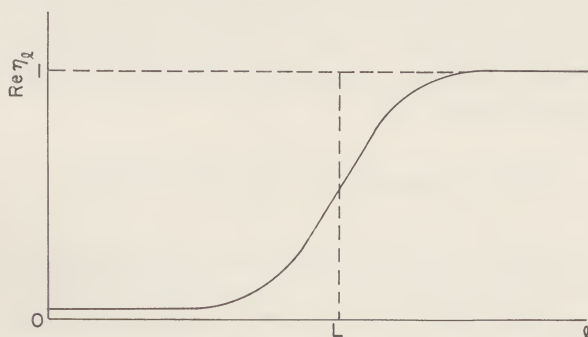


Fig. 2. Typical behavior of  $\text{Re } \eta_l$  for strong absorption cases.  $L$  is the critical angular momentum.

Typically, for strong absorption, we have the qualitative behavior shown in Fig. 2 of  $\text{Re } \eta_l \equiv \text{Re } e^{2i\delta_l}$ , where  $\delta_l$  is the complex phase shift for elastic scattering. The  $\text{Im } \eta_l$  tends to be small but can be included in the following discussion. The behavior shown can be, and often has been, approximated by the sharp cut off,

$$\text{Re } \eta_l = \begin{cases} 0 & \text{for } l < L \\ 1 & \text{for } l > L, \end{cases} \quad (2)$$

but this approximation is not necessary. What is needed to simplify the treatment is, for instance, that  $\text{Re } \eta_l$  have the same characteristics for the incident and outgoing channels. The behavior of  $\eta_l$  is, further-

more, obtainable directly from elastic and total reaction cross sections. We shall use a W.K.B. treatment, similar to that used by Gottfried and Jackson [12]. For simplicity we shall again neglect spins in the incident and final channels. It is known that in nuclear applications the spin of the target can be neglected, although spin-orbit effects for the incident and emerging particles may be important. At high energies both projectile and target spins may matter. The derivation can be generalized by way of the helicity amplitudes [13] or by other means [14]. Our starting point is the DWBA matrix element [15]

$$M_{fi} = \langle \Psi_{k_f}^- | H' | \Psi_{k_i}^+ \rangle, \quad (3)$$

where  $H'$  is the transition operator or potential, which is assumed to contribute only a small fraction to the absorption in the incident or final channel. The superscripts,  $-$  and  $+$ , refer to ingoing and outgoing wave boundary conditions, respectively. With use of the high energy W.K.B. approximation [16]

$$\Psi_{\mathbf{k}}^{\pm}(\mathbf{r}) = \Phi \exp \left[ i\mathbf{k} \cdot \mathbf{r} - \frac{i}{v} \int_0^{\infty} U^{\pm}(\mathbf{r} \mp \hat{\mathbf{k}}s) ds \right], \quad (4)$$

where  $\Phi$  represents internal (bound) wavefunctions, such as those of the target and final nuclei and  $U^{\pm}$  is the complex optical potential in the initial or final state. If  $H'$  is of short range, or local, then the relevant coordinates in the initial and final state become identical. If, as suggested by the earlier model, we take the  $z$ -axis along  $\mathbf{k}_i + \mathbf{k}_f$ , then integration over all internal coordinates gives

$$M_{fi} = \int d^3r e^{i\mathbf{q} \cdot \mathbf{r}} \phi_f(\mathbf{r}) \times \\ \times \exp \left[ -\frac{i}{v_i} \int_{-\infty}^z U^+(x, y, z') dz' - \frac{i}{v_f} \int_z^{\infty} U^-(x, y, z') dz' \right], \quad (5)$$

where  $\mathbf{q}$  is the three momentum transfer,  $\mathbf{k}_i - \mathbf{k}_f$ , and  $\phi_f(\mathbf{r})$  is the transfer factor, which remains after the internal coordinate (if any) integration has been performed. For deuteron stripping reactions, for instance, this is the  $n$ - $p$  potential strength multiplied by the wavefunction of the captured nucleon, whereas for a peripheral interaction at high energy it is the relevant coupling strengths multiplied by the spatial

representation of the propagator. These transfer factors are illustrated in Fig. 3.

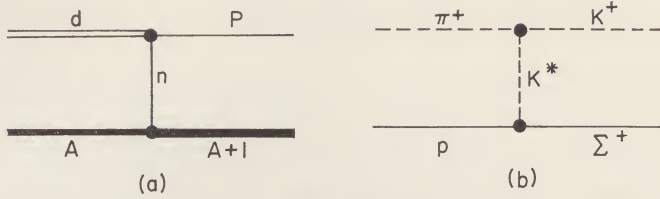


Fig. 3. Feynman diagrams for direct stripping reaction (a) and peripheral production of  $K^+\Sigma^+$  system (b). The transfer factor,  $\phi_j$ , is related to the  $n$  and  $K^*$  in these two processes.

There are at least three cases in which Eq. (5) can be related to the elastic scattering amplitude [6],

$$(e^{2i\delta_l} - 1) = \frac{k^2}{2\pi i v} \int_{-1}^1 P_l(\cos \theta) d(\cos \theta) \int e^{-i\mathbf{k}_r \cdot \mathbf{r}} U(\mathbf{r}) \psi_{\mathbf{k}_i}^+(\mathbf{r}) d^3r, \quad (6)$$

where  $\theta$  is the scattering angle,  $\delta_l$  is the phase shift of angular momentum  $l$  and  $\psi_{\mathbf{k}_i}^+$  is the elastic scattering wave-function for the optical potential  $U$ . The first of these cases occurs when  $U^+ \approx U^-$  and  $k_i \approx k_f$ ,  $v_i \approx v_f$ , as is often true when strong absorption is present in both initial and final channels. The second case occurs when  $\phi_j(\mathbf{r})$  is of short range, and the third one when the main contribution occurs at  $z \approx 0$  (ring locus). In all cases we can write

$$M_{fi} = \sum_l (2l+1) e^{i\delta_l^-} B_l(k, j) e^{i\delta_l^+} P_l(\cos \theta), \quad (7)$$

where in the first instance  $\delta_l^- \approx \delta_l^+$ . These phase shifts are given by

$$2\delta_l^\pm = -\frac{1}{v_{(f)}} \int_{-\infty}^{\infty} U^\pm(x, y, z') dz', \quad (8)$$

where the  $z'$  axis is taken along  $\mathbf{k}_i + \mathbf{k}_f$  and the identification  $k(x^2 + y^2)^{\frac{1}{2}} = l$  is to be made. The amplitude  $B_l$  is the angular momentum representation of the Born approximation transition amplitude

$$B_l(k, j) = \frac{1}{2} \int_{-1}^1 d(\cos \theta) P_l(\cos \theta) \int \phi_j(\mathbf{r}) e^{i\mathbf{q} \cdot \mathbf{r}} d^3r \quad (9)$$

with the same  $z$ -axis as for Eq. (8). In terms of  $M_{fi}$ , the differential

cross section for a two particle reaction is given by

$$\frac{d\sigma}{d\Omega} = \frac{k_f^2}{4\pi^2 v_i v_f} |M_{fi}|^2. \quad (10)$$

For a sharp transition, Eq. (2), we recover the sharp cut-off model. However, the most important asset of Eq. (7) is that it allows us to make direct use of elastic scattering results. The phase shifts  $\delta_l^\pm$  can be obtained directly from experiment. The difference from a full DWBA treatment, is that there is no need to determine an optical potential which fits the scattering data, and of further using this potential to calculate distorted waves. These intermediate steps are bypassed and the elastic scattering phase shifts are used directly. Thus, when the W.K.B. approximation is valid, Eq. (7) can be justified. In this form, but with spin generalizations where required, it has been applied to numerous peripheral reactions [3, 12], including charge exchange processes, at high energies.

The cross section generally falls off rapidly with increasing angle because the contribution of the small angular momenta is reduced by the absorption (see Fig. 2). For peripheral or surface reactions, most of the contributions to  $M_{fi}$  arise from a small band of angular momenta,  $l$ , around  $L \approx kR - \frac{1}{2}$ . Eq. (7) can also be applied to nuclear processes [11, 14], including inelastic scattering. In fact, we can recover the adiabatic approximation limit of Austern and Blair [5]. In this instance, we write out the explicit dependence of the optical potential on the nuclear radius  $R$ , namely  $U(r, R)$ . For surface mode excitations, the radius  $R$  can be written in terms of an average spherical radius  $R_0$  as  $R = R_0 + \alpha$ . The parameter  $\alpha$  may depend on angle and is a measure of the surface deformability. To lowest order, the interaction potential is then

$$\Delta U \approx \alpha \left. \frac{\partial U(R, r)}{\partial R} \right|_{\alpha=0}.$$

The form factor,  $\phi_j$ , is the expectation value,  $\bar{\alpha}$ , of  $\alpha$  between initial and final states, multiplied by  $\partial U / \partial R|_{\alpha=0}$ . By using Eqs. (6) and (8) with the exact potential  $U_0 + \Delta U = U$ , and comparing it to Eq. (7) we find to lowest order in  $\Delta U$



$$B_l \approx \frac{\pi}{k^2} \int_{-\infty}^{\infty} \Delta U(x, y, z) dz = \frac{i\pi v}{k^2} \frac{\bar{\alpha}}{\eta_l} \frac{\partial \eta_l}{\partial R} \Big|_{\alpha=0}. \quad (11)$$

This follows because  $\partial \eta_l / \partial R = 2i\eta_l (\partial \delta_l / \partial R)$  and it has been assumed that  $k_i \approx k_f \approx k$ . This expression can be compared to the expression derived by Austern and Blair [5] and shows their result can be obtained when the W.K.B. approximation is valid.

We see that the impact parameterization or angular momentum decomposition allows a direct comparison to be made with the elastic scattering in a rather simple and visualizable manner. It has proven its usefulness both in nuclear and high energy physics.

#### 4. EXTENSIONS TO THREE PARTICLE FINAL STATES

The diffraction models developed in the last section and the comparison to elastic scattering may also be useful when three particle final states occur. This is true, in particular, when resonances or strong attractive interactions are present between two of the final state particles. The optical potential then acts primarily on the center-of-mass of these two interacting particles, as is normally assumed for a bound system and not on the internal (relative motion) coordinates. For instance, the direct reaction  $A(\text{He}^3, pp)A'$ , where  $A$  is an arbitrary target nucleus and  $A'$  the final nucleus, may be analyzable in this way. The optical potential only affects the center-of-mass motion of the two protons when they are closely correlated (small relative momentum). In addition, if the quantum numbers and particularly the energy of the resonance or virtual state are not very different from those of a bound system, then the optical potential for the latter may be taken to be comparable to the former. For the above example the deuteron serves as such a bound state, despite its different spin quantum numbers, since the absorption parameters of the optical potential do not appear to depend strongly on this spin. Because of the spin isospin difference of the  $pp$  and  $d$ , one finds [18]

$$\frac{d^2 \sigma(\text{He}^3 + A \rightarrow p + p + A') / d\Omega dE}{d\sigma(\text{He}^3 + A \rightarrow d + A'') / d\Omega} = (\text{phase space factor}) \times \\ \times (\text{spin-isospin factor}) \times \left| \frac{\int \psi_{\text{He}}(\mathbf{x}) \chi_a^{-*}(\mathbf{x}) d^3 x}{\int \psi_{\text{He}}(\mathbf{x}) \psi_d(\mathbf{x}) d^3 x} \right|^2, \quad (12)$$

where  $\psi_{\text{He}}(x)$  is the internal spatial wavefunction of  $\text{He}^3$  integrated over one relative coordinate,  $\chi_{q'}^-$  is the scattering pp wavefunction for relative momentum  $q'$  with incoming waves and  $\psi_d$  is the deuteron wavefunction. This ratio is *independent* of the absorption and allows a direct comparison to the  $(\text{He}^3, d)$  reaction. This method should be equally applicable for high energy peripheral processes. For instance, a relation like Eq. (12) may hold for the ratio

$$\frac{\sigma(\pi^+ + p \rightarrow Y_1^{*+}(1385 \text{ MeV}) + K^+)}{\sigma(\pi^+ + p \rightarrow \Sigma^+ + K^+)} \quad (13)$$

$\downarrow \rightarrow A + \pi^+$

if the absorbing potential is mainly due to other channels. The two peripheral reactions which appear in the ratio (13) can be mediated by  $K^*$  exchange (see Fig. 3). It thus follows from Eq. (12) that information on the internal resonance or virtual state may be obtainable, independent of the wavefunction distortions in the incident and final states. These ideas are only beginning to be investigated. For high energy reactions, furthermore,  $\text{SU}_3$  and higher symmetries may be used to simplify the analyses of 3-body reactions, but this is not relevant for the ratio (13).

## CONCLUSIONS

In the absence of detailed optical potential parameters, the diffraction model in its crude or more refined form serves as a useful guide to understanding reaction differential cross sections, as well as polarizations and decay correlations when spins are included. The conditions for its applicability may be met in both medium energy nuclear physics and high energy particle reactions.

I would like to thank Dr. J. S. Blair and Dr. I. Halpern for helpful comments.

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# INTUITIVE ANALYTICITY

GUNNAR KÄLLÉN

*Department of theoretical physics, University of Lund, Lund, Sweden*

*(Received April 30, 1965)*

## 1. INTRODUCTION

During the last fifteen years the concept of analytic functions has become very important in elementary particle physics in general and field theory in particular. Such functions have been used both for phenomenological fittings of data and as rigorous mathematical tools in the proofs of general theorems. Evidently, these two fields of application are not sharply distinct. The transition region consists essentially of those "dispersion relations" which can both be proved from general field theoretical principles and be used for an analysis of actual measurements. However, it must be admitted that the class of phenomena where this takes place is rather limited and consists essentially only of forward dispersion relations for  $\pi$ -meson nucleon scattering. The philosophy of scientists working with analytic functions range from the very strict mathematical position that only those analyticity properties which can actually be proved with absolute rigour from the foundations of field theory should be used to the other extreme, viz. that the whole formalism of field theory should be abandoned in favour of the concepts of "maximal analyticity" and unitarity. Although none of these two viewpoints can claim to have achieved a great success or a general break through as far as our understanding of the physics of elementary particles is concerned, there seems to be a general agreement that analytic properties of various functions are of interest. The uninitiated reader who is not a disciple of either of the two schools mentioned above usually has difficulties in reading the relevant papers. The strict mathematical arguments are often extremely technical and not very transparent. The phenomenological papers are sometimes based on rather drastic assumptions, the physical significance of which is very difficult to look through. The position we want to adopt in this note is neither one nor the other of the two extremes



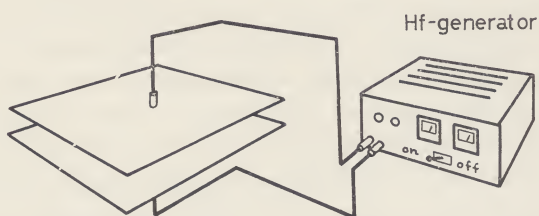
mentioned above. Rather, we will try to discuss, on a very low level of mathematical rigour, what we feel to be the physical significance of the analyticity concept in field theory and elementary particle physics. For this purpose, we want to consider just one example. To limit the necessary mathematical machinery as much as possible we shall consider only vacuum polarization and the "two point function". The main point that emerges at the end of our discussion is the realization that *the analytic continuation of a physical quantity off the real axis and into the complex plane can be understood as a substitute for the averaging over space and time* which, according to a classical paper by Bohr and Rosenfeld, is the physical basis of field theory. It has very often been suggested that the concept of a field which is based on an idealization of a classical measurable quantity with the aid of macroscopic test bodies has no relevance for elementary particle physics and that analyticity is a reasonable substitute for the field concept itself. We want to emphasize that analyticity is, indeed, a substitute for the field concept, but in such a way that as soon as we have analytic functions and want to discuss seriously what happens out in the complex plane, we automatically use something which can be described as a field smeared with the aid of a classical space time measurement. Even if this point is not absolutely new and can be found in the literature [7], we still believe it is not generally recognized and that it might be worth while to point it out once more.

## 2. VACUUM POLARIZATION IN QUANTUM ELECTRODYNAMICS

To illustrate how analytic functions enter into field theoretical calculations, we discuss an elementary and wellknown problem, viz. the vacuum polarization in an external electromagnetic field. Even if this phenomenon is of some practical importance, e.g., in the Lamb shift calculation, in proton-proton scattering and in  $\mu$ -particle atoms to mention a few examples, we here prefer to discuss a "Gedankenexperiment" to bring out the general idea. For this purpose, we imagine that we have a large classical condenser connected to a high frequency generator in the way indicated in Fig. 1. Using classical electromagnetic theory it is possible to calculate the electromagnetic field between the condenser plates caused by the high frequency generator. To simplify the discussion as much as possible, we assume that the

condenser plates are sufficiently wide and the distance between them sufficiently narrow for the classical electromagnetic radiation from the condenser to be neglected. The field that actually exists between the condenser plates can be measured with the aid of a charged classical test body introduced in the space between the plates. If this measurement could be performed with a very high accuracy, one would find that the actual field observed between the plates is not the same as the

Condenser



field which one calculates using classical electromagnetic theory. The difference is due to the fact that the electromagnetic field “polarizes the vacuum” by creating (virtual or real) electron-positron pairs between the plates and those pairs generate a non-vanishing electromagnetic field. The quantity actually measured by the test body is the sum of the applied external field and the field from the electron-positron pairs. If the applied external field is weak enough, we expect the polarization field to be linear in the external field. The analogy between this situation and the polarization effects in a classical dielectric medium between the condenser plates needs hardly be emphasized. For mathematical convenience let us consider the electric current charge distribution between the plates instead of the fields. The two quantities are evidently equivalent as one can be obtained from the other through the classical Poisson equation. Both in the case of a classical dielectric medium and in the case of the quantum mechanical vacuum polarization effect we have a linear relation between the induced current  $\delta j_\mu(x)$  and the applied external current  $j_\mu^{\text{ext}}(x)$  of the form

$$\delta j_\mu(x) = \int dx' K(x-x') j_\mu^{\text{ext}}(x'). \quad (1)$$

The kernel function  $K(x-x')$  in Eq. (1) plays the role of a “dielectric constant”, both in the classical case and in quantum electrodynamics.

We are going to refer to it as the “dielectric constant of the vacuum”. For invariance reasons, this function depends essentially only on the square of the vector  $x-x'$ . The word “essentially” here should be understood to mean that the function  $K$  depends on this quantity except for the fact that it might have different values in the forward and in the backward light cone \*. As a matter of fact this reservation has to be made because of the requirement that Eq. (1) is “causal” in structure. An important physical condition is, of course, that the polarization current  $\delta j_\mu(x)$  is identically zero for any time prior to the time when the external current  $j_\mu^{\text{ext}}(x)$  from the high frequency generator is switched on. In mathematical language, this means the requirement

$$K(x-x') = 0 \quad \text{for } x_0 < x'_0. \quad (2)$$

Further, Eq. (2) has to hold in all coordinate systems, i.e., for all space like separations between the points  $x$  and  $x'$ . It follows that we can write

$$K(x-x') = 0 \quad \text{for } (x-x')^2 > 0 \text{ and for } x_0 < x'_0. \quad (3)$$

Consequently, the function  $K(x-x')$  is an expression which vanishes identically in a rather large domain in  $x$ -space. As is well-known, such a function has a Fourier transform with certain analyticity properties. For the particular case under discussion here, we can write

$$K(x-x') = \frac{1}{(2\pi)^4} \int dp e^{ip(x-x')} \Pi_R(p^2), \quad (4)$$

$$\Pi_R(p^2) = \bar{\Pi}(p^2) + i\pi \frac{p_0}{|p_0|} \Pi(p^2) + \text{renormalization terms.} \quad (4a)$$

The function  $\Pi_R(p^2)$  is referred to here as the “retarded polarization function”. The terminology “retarded” clearly comes from the fact that the function  $K(x-x')$  vanishes as indicated in Eq. (3). Equation (4a) splits the retarded polarization function in a real and an imaginary part. In general, it is to be expected that the Fourier transform of the dielectric constant as defined in Eq. (4) should be a complex number.

\* In technical language, this means that  $K$  is invariant only under Lorentz transformations not involving time reflections.

This is true both in the classical case and in quantum electrodynamics. The imaginary part of such a frequency dependent dielectric constant is related to a possible energy dissipation in the system. Classically, the imaginary part corresponds to the losses in the dielectric medium because of the work necessary to change the polarization of the molecules. Physically, these losses make their appearance in a heat dissipation in the condenser. Quantummechanically, the imaginary part corresponds to the possible creation of real pairs which leave the condenser and, therefore, do not belong to the system under consideration any more. The analytic properties referred to above turn out to be that the function  $\Pi_R(p^2)$  is an analytic function of  $p^2$  regular for all complex values of  $p^2$  as well as for spacelike values of  $p^2$ . Introducing the variable  $z$  defined by

$$z = (p_0 + i\varepsilon)^2 - \vec{p}^2 \sim -p^2, \quad (5)$$

we can understand the function  $\Pi_R(p^2)$  as the boundary value of an analytic function  $\Pi_R(z)$  regular everywhere except on the positive real axis. Using standard Cauchy techniques one then finds that the real and imaginary parts of the boundary value of this function are related through a "Hilbert transform", viz.

$$\overline{\Pi}(p^2) = \int_0^\infty \frac{da \Pi(-a)}{(a + p^2)_P} \quad (6)$$

where  $P$  in the denominator indicates that the principal value has to be taken. Further, the complete analytic function  $\Pi_R(z)$  can be written in the form

$$\Pi_R(z) = \int_0^\infty \frac{da \Pi(-a)}{a - z} + \text{renormalization terms.} \quad (7)$$

Equations (4)–(7) are an example of what is normally referred to as a "dispersion relation". Indeed, for the case of a classical dielectric medium, the corresponding relation which looks nearly identical with the equation written out here was the first important case where a dispersion relation was discussed [1]. Also in quantized field theories this was actually the first case where dispersion relations were explicitly given [2]. We have rederived it here, using essentially standard arguments. However, one point, in particular, should be emphasized



here. One very often finds the statement in the literature that dispersion relations are based on the condition that commutators between various operators vanish for spacelike separations. It should be noted that the argument above contains no reference at all to this point, but only the "classical" condition that the kernel  $K(x-x')$  appearing in Eq. (1) is "causal", i.e., that it vanishes for all points such that  $x$  is earlier than  $x'$ . This is the basic condition which yields analytic properties for various functions both in classical physics and in field theory.

### 3. THE VACUUM POLARIZATION KERNEL AND THE COMMUTATOR CONDITION

As was mentioned at the end of the last section, the analytic properties of the Fourier transform of the vacuum polarization kernel  $K(x-x')$  follow from the causal condition stated in Eq. (2). However, because of Lorentz invariance this condition is immediately generalized to the stronger restriction given in Eq. (3), i.e., that the  $x$ -space kernel vanishes not only when  $x_0$  is earlier than  $x'_0$  but also for all spacelike distances  $x-x'$ . As a matter of fact, this last requirement which follows from the combined conditions of causality and Lorentz invariance is related to the vanishing of a commutator of field operators for space-like separations. To discuss this in more detail, we remark that the kernel  $K$  can be related to a vacuum expectation value of a product of two current operators with the aid of a "reduction formula" in the following way [3]:

$$3\Box K(x-x') = i\theta(x-x')\langle 0|[j_\mu(x), j_\mu(x')]|0\rangle + \text{renormalization terms}, \quad (8)$$

$$\theta(x-x') = \frac{1}{2} \left( 1 + \frac{x_0 - x'_0}{|x_0 - x'_0|} \right) = \begin{cases} 1 & \text{for } x_0 > x'_0 \\ 0 & \text{otherwise} \end{cases} \quad (8a)$$

One sees immediately from the right hand side of Eq. (8) that the explicit step function  $\theta(x-x')$  guarantees the classical causality in Eq. (2) while the stronger condition in Eq. (3) which is requested by Lorentz invariance implies that the commutator between the two current operators on the right hand side of Eq. (8) has to vanish for spacelike separations [4]. The renormalization terms not written out explicitly in Eqs. (4a) and (8) do not influence this conclusion as they are of a point interaction character, i.e., they are proportional to a

$\delta$ -function or, rather, derivatives of a  $\delta$ -function at the point  $x = x'$ .

The imaginary part of the Fourier transform  $\Pi(p^2)$  in Eq. (4a) can be separated out from Eq. (8). It is essentially given by the right hand side but with the step function suppressed. More explicitly, one has

$$\langle 0 | j_\mu(x) j_\mu(x') | 0 \rangle = \frac{-3}{(2\pi)^3} \int dp e^{ip(x-x')} p^2 \Pi(p^2) \theta(p). \quad (9)$$

Equation (9) here is quite interesting as the momentum space expression appearing on the right hand side has a structure which is somewhat similar to the configuration space expression on the right hand side of Eq. (8). In particular, the function under the integral sign in Eq. (9) vanishes unless  $p$  has a positive time component. For invariance reasons it must therefore also vanish for spacelike values of  $p^2$ . Physically, this is quite reasonable because the imaginary part of the dielectric constant corresponds to the creation of real pairs in the experimental apparatus of Fig. 1. The vector  $p$  is the total energy momentum vector of the real particles which are created by the external field. Consequently, this vector must be timelike. Using standard arguments it then follows that the left hand side of Eq. (9) can be extended to an analytic function in configuration space. Indeed, one has

$$\langle 0 | j_\mu(x) j_\mu(x') | 0 \rangle = \text{boundary value of } F(z), \quad (10)$$

$$F(z) = -3i \int_0^\infty da a \Pi(-a) \Delta^{(+)}(z, a), \quad (10a)$$

$$\Delta^{(+)}(z, a) = \frac{-i}{(2\pi)^3} \int dp e^{ip(x-x')} \delta(p^2 + a) \theta(p) = \frac{-a}{8\pi} \frac{H_1^{(1)}(\sqrt{az})}{\sqrt{az}}, \quad (10b)$$

$$z = (x_0 - x'_0 - i\epsilon)^2 - (\vec{x} - \vec{x}')^2 \sim -(x - x')^2. \quad (10c)$$

#### 4. PHYSICAL INTERPRETATION OF THE $x$ -SPACE FUNCTIONS $F(z)$ .

At the first moment, the construction at the end of the last section appears to be very formal and mathematical. Off hand, it is not easy to give any physical meaning to the function  $F(z)$  for complex values of the variable  $z$  in Eq. (10c). The normal terminology is also that real values of the variable  $z$  correspond to "physical points" while points out in the complex plane are referred to as "the unphysical region". We shall try to make the point here that this terminology is quite mis-

leading and that the reverse nomenclature would really be more appropriate. To substantiate this somewhat paradoxical statement we remark that, according to well-known principles in quantum field theory, the really observable quantity is not the field as such but, rather, a space time average of the field. The physical background for this goes back, as has already been mentioned in the introduction, to an old paper by Bohr and Rosenfeld [5]. More formally, this means that the really observable quantities are expressions of the form

$$j_\mu(f) = \int dx f(x) j_\mu(x), \quad (11)$$

where  $f(x)$  is a “test function”. It is a smoothly varying function which is appreciably different from zero only inside a small space time region. The extension of this function in space corresponds to the extension of the classical test bodies which are used to measure the current distribution  $j_\mu(x)$  while its extension in time corresponds to the time interval which is necessary to perform the measurement. For our purpose, it is of particular interest to consider the following test function

$$f(x) = \frac{1}{\pi} \frac{\alpha}{\alpha^2 + (x_0 - T)^2} \delta(\bar{x} - \bar{X}). \quad (12)$$

Here, we have permitted ourselves to use the idealization that the test body has no extension in space while the measurement of the current is supposed to be performed during a time interval of the order of magnitude  $\alpha$ . In the limit when  $\alpha$  goes to zero, the test function in Eq. (12) becomes a  $\delta$ -function also in the time coordinate. Using two test functions of this particular kind, one centered around the point  $(\bar{X}, T)$  and with a time smearing  $\alpha$  and the other centered around the point  $(\bar{X}', T')$  and with time smearing interval  $\alpha'$ , one can substitute in Eq. (10) and perform the integrations over  $x$  and  $x'$ . After a straight forward calculation, one finds

$$\begin{aligned} \langle 0 | j_\mu(f) j_\mu(f') | 0 \rangle &= \\ &= \frac{\alpha \alpha'}{\pi^2} \iint \frac{dx_0 dx'_0}{[\alpha^2 + (x_0 - T)^2][\alpha'^2 + (x'_0 - T')^2]} \langle 0 | j_\mu(\bar{X}, x_0) j_\mu(\bar{X}', x'_0) | 0 \rangle \\ &= -3i \int da a \Pi(-a) \Delta^{(+)}(z, a) = F(z), \end{aligned} \quad (13)$$

$$z = [T - T' - i(\alpha + \alpha')]^2 - (\bar{X} - \bar{X}')^2. \quad (13a)$$

The characteristic feature of the result exhibited in Eq. (13) is that we obtain the analytic function defined in Eq. (10a) evaluated at a point out in the complex plane and not at the real axis. The imaginary part of the variable  $z$  in Eq. (13a) is essentially determined by the sum of the two smearing parameters  $\alpha$  and  $\alpha'$ . In this way, we see that the use of the analytic function  $F(z)$  out in the complex plane and not of its boundary value can be thought of as a replacement for the smearing over test functions  $f(x)$  which should always be used if we want to discuss measurable quantities in field theory. In fact, we see that it is impossible to reach the real axis without making the idealization that the time smearing interval  $\alpha$  can be put equal to zero. Strictly speaking, this is an unallowed idealization and, in this sense, the real physical points are the points out in the complex plane and not the points on the (positive) real axis.

The discussion above contains the idealization that the spatial extension of the test body is neglected. From the point of view of physics this is not allowed. However, if we replace the three-dimensional  $\delta$ -function in Eq. (12) by a smoothly varying function which is different from zero only in the neighborhood of the point  $\bar{X}$ , this only changes the result (13) to contain an average of the function  $F(z)$  in a neighborhood of the complex point  $z$  but still far away from the real axis. By suitable adjustments the domain over which the averaging is made can be made arbitrarily small. Therefore, it appears that the averaging in time is more essential than the averaging over the space coordinates [6].

## 5. CONCLUDING REMARKS

The discussion above has purposely been made very simple and elementary. The main purpose has been to illustrate how the analyticity concept which plays a significant role in some modern approaches to quantized field theory can be understood and interpreted on an intuitive level. To simplify the discussion as much as possible we have restricted ourselves to one particular phenomenon, viz. vacuum polarization in quantum electrodynamics and one mathematical expression, viz. the vacuum expectation value of a product of two opera-



tors. However, it should be reasonably clear from the argument above that both the methods and the results are not limited to this particular problem. Especially the smearing process discussed in section 3 can, as well, be applied to a product of  $n$  operators as to a product of 2 operators. Also, the main remark of the discussion in sections 1 and 2, viz. that the analyticity properties in  $p$ -space follow essentially from the retarded character of the corresponding  $x$ -space functions and that the vanishing of the commutator is necessary to guarantee Lorentz invariance but does not really correspond to the basic analyticity requirements, can be applied to the more complicated case of a product of  $n$  operators. However, the formal mathematics gets more and more involved the higher the value of  $n$  and we do not want to enter upon the rather intricate mathematical formalism which is necessary to discuss the general case.

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- 3) For proof, see e.g. G. Källén, Helv. Phys. Acta **25** (1952) 417 or Handbuch der Physik V<sub>1</sub>, (Springer, 1958).
- 4) For the more complicated case of forward dispersion relations for  $\pi$ -meson nucleon scattering a somewhat similar situation occurs. The actual analyticity properties of the scattering amplitude come because of an explicit step function of the same kind as in Eq. (8a) while the vanishing of the commutator between two "meson currents" is necessary to guarantee relativistic invariance and certain boundedness properties. Cf., e.g., the discussion in G. Källén, Elementary Particle Physics, (Addison-Wesley, 1964) esp. pp. 138–139.
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H. J. Borchers, Nuovo Cim. **33** (1964) 1600.
- 7) Cf. e.g. L. Rosenfeld, Nucl. Phys. **26** (1961) 579 esp. p. 580, and other authors quoted there. This ref. discusses nuclear reactions where a similar situation arises.

# A NOTE ON BARYON MASSES, MASS DIFFERENCES AND MAGNETIC MOMENTS, ACCORDING TO VARIOUS SYMMETRY SCHEMES \*

BERNARD T. FIELD

*Massachusetts Institute of Technology, Cambridge, Massachusetts*

*(Received May 3, 1965)*

## INTRODUCTION

The material presented herein is not new; the results have all appeared in the literature, derived in most cases by elegant application of group theoretical techniques. Nevertheless, the possibility of understanding the relationships among the static properties of the baryons in terms of relatively simple physical ideas, following from the assumed symmetry properties of the fundamental interactions among the "elementary" particles, may be of more than pedagogic value providing, as it does, a concrete physical model in terms of which such symmetry schemes can be visualized by use of concepts which have become, through usage, part of the stock-in-trade of most practicing physicists.

The examples considered in this note will be confined to three such symmetry schemes, namely:

(1) The doublet model, suggested by Schwinger and developed by him, by Pais, Gell-Mann, Sakurai, and many others, in which the basic symmetry gives rise to four isotopic doublets of baryons: the nucleons  $N = (n, p)$  of hypercharge  $Y = 1$ ; two  $Y = 0$  doublets,  $Y = (\Sigma^+, Y^0 = \sqrt{\frac{1}{2}}(\Sigma^0 - \Lambda^0))$  and  $Z = (Z^0 = \sqrt{\frac{1}{2}}(\Sigma^0 + \Lambda^0), \Sigma^-)$ ; the  $Y = -1$  cascade doublet,  $\Xi = (\Xi^0, \Xi^-)$ . The interaction with the  $\pi$ -meson field is such as to maintain the complete degeneracy of these doublets. However the K-meson interaction removes this degeneracy, causing

\* Considering that some of the ideas expressed in this note were developed during the author's stay at CERN, in the academic year 1960, and that the approach owes much of its inspiration to the example consistently set by Viki Weisskopf, of understanding complex phenomena in terms of elementary ideas, it is a pleasure and a privilege to dedicate this modest opus to V. F. Weisskopf.

the  $Y = 0$  baryons to split into a singlet,  $\Lambda^0$ , and a triplet  $\Sigma = (\Sigma^+, \Sigma^0, \Sigma^-)$ . In addition, the interaction causes the masses of these particles to be different, as well as a separation of the masses of the N and  $\Xi$  doublets.

(2) The Fermi-Yang-Sakata model, in which all the observed particles (mesons and baryons) are regarded as compound states of a basic baryon-triplet (p, n,  $\Lambda^0$ ). The fundamental interaction is such as to be invariant with respect to the interchange of any two members of the triplet.

(3) The octet model of Gell-Mann and Ne'eman, in which the basic symmetry among the baryons derives from the eight-fold representation of the group SU(3), thereby including the eight observed baryons: the N-doublet ( $Y = 1$ ), the  $\Lambda^0$ -singlet and  $\Sigma$ -triplet ( $Y = 0$ ) and the  $\Xi$ -doublet ( $Y = -1$ ).

#### BARYON MASS FORMULAE

(1) A charge independent interaction among the baryon doublets through the K-meson field has the effect of mixing the  $Y = 0$  doublets; in addition, the possibility of a K-meson interaction which depends linearly in the hypercharge,  $Y$ , has the effect of splitting the mass degeneracy between the N- and  $\Xi$ -doublets. Such an interaction can be described in terms of the effective Hamiltonian

$$H_K = AY + B\tau_Y \cdot \tau_Z.$$

The eigenstates of this Hamiltonian are N,  $\Lambda^0$ ,  $\Sigma$  and  $\Xi$ , with the masses (total energies) becoming

$$\begin{aligned} M_N &= M_0 + a \\ M_\Xi &= M_0 - a \\ M_\Sigma &= M_0 + \frac{1}{4}b \\ M_\Lambda &= M_0 - \frac{3}{4}b \end{aligned} \tag{1}$$

from which follows the relationship

$$M_0 = \frac{M_N + M_\Xi}{2} = \frac{3M_\Sigma + M_\Lambda}{4}. \tag{2}$$

Substituting the known masses

$$1128.2 \stackrel{?}{=} 1173.5 \text{ MeV,}$$

a difference of 45.3 MeV. Although this discrepancy is small as compared to  $M_0$  ( $\approx 4\%$ ), it is appreciable when measured in terms of the strength of the symmetry-breaking interaction ( $a = -189.3$  MeV,  $b = 77.5$  MeV).

(2) The Fermi-Yang-Sakata model encounters severe difficulties in attempts to derive the properties of the other observed baryons,  $\Sigma$  and  $\Xi$ . One possibility is to consider these as compounds of two members of the triplet plus one anti-particle, i.e.  $\Sigma = N\bar{N}\Lambda$ ,  $\Xi = \Lambda\bar{N}\Lambda$ . Considering such a model in analogy with linear triatomic molecules, Yamaguchi was able to choose interaction constants such as to satisfy the observed mass differences; however, this model leads to the prediction of two new compound baryons with masses comparable to the others,  $N' = N\bar{N}N$  and  $X = N\bar{\Lambda}N$  (strangeness = +1, or  $Y = 2$ ), neither of which exists in nature in the mass range predicted. Furthermore, according to this model, the compound baryons should all have spin  $\frac{1}{2}^-$ .

A more sophisticated model starts with the  $(p, n, \Lambda^0)$  triplet as a 3-fold representation of the group  $SU(3)$ . The combination of a baryon and an anti-baryon then gives rise to nine mesons, as per the following table:

TABLE 1  
Mesons obtained from the Fermi-Yang-Sakata Model

$Y$	$I$	$I_3$	Combination	Particle
0	0	0	$\sqrt{\frac{1}{3}}(p\bar{p} + n\bar{n} + \Lambda\bar{\Lambda})$	$X^0(?) \omega_0$
0	0	0	$\sqrt{\frac{1}{6}}(p\bar{p} + p\bar{n} - 2\Lambda\bar{\Lambda})$	$\eta \quad \phi_0$
0	1	1	$pn$	$\pi^+ \quad \rho^+$
		0	$\sqrt{\frac{1}{2}}(p\bar{p} - n\bar{n})$	$\pi^0 \quad \rho^0$
1	$\frac{1}{2}$	-1	$n\bar{p}$	$\pi^- \quad \rho^-$
		$\frac{1}{2}$	$p\bar{\Lambda}$	$K^+ \quad K^{*+}$
		$-\frac{1}{2}$	$n\bar{\Lambda}$	$K^0 \quad K^{*0}$
-1	$\frac{1}{2}$	$\frac{1}{2}$	$\Lambda\bar{n}$	$\bar{K}^0 \quad \bar{K}^{*0}$
		$-\frac{1}{2}$	$\Lambda\bar{p}$	$K^- \quad K^{*-}$

These do correspond to the observed pseudo-scalar meson octet and to the vector meson singlet and octet.



However, the next step, that of obtaining the other observed baryons by the combination of a basic triplet of baryons and an octet of mesons does not lead to the observed particles, since the combinations

$$3 \otimes 8 = 3 \oplus 6 \oplus 15 \quad (3)$$

none of which multiplets corresponds to any of the observed baryon groups. \*

The difficulty here, as is well known, arises from the fact that the baryon triplet is not symmetrically placed with respect to the  $Y$  vs  $I_3$  (or  $Q$  vs  $U_3$ ) axes. Many schemes have been suggested for overcoming this difficulty, of which the most popular at the moment is the one in which a symmetrical triplet of baryons whose charges are multiples of  $1/3e$  (the quarks of Gell-Mann) is substituted for the Fermi-Yang-Sakata triplet. Since most of the consequences of these models, as far as the known baryons are concerned, are the same as those obtained by starting with a baryon octet, we shall confine our attention to the latter model.

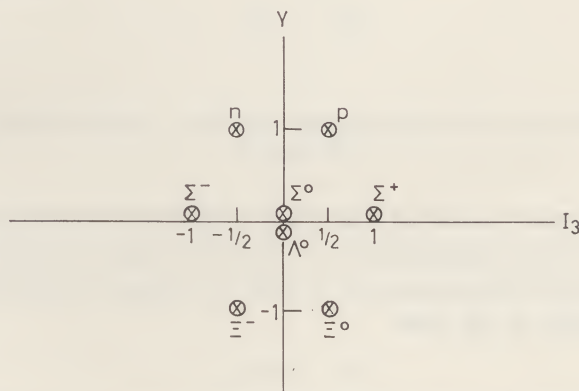


Fig. 1.  $Y$  vs  $I_3$  for the baryon octet according to  $SU(3)$ .

(3) In the octet representation of  $SU(3)$ , the eight observed baryons are symmetrically placed with respect to the  $Y-I_3$  axes, with one place ( $Y = 0, I_3 = 0$ ) being occupied by both the  $\Sigma^0$  ( $I = 1, I_3 = 0$ ) and

\* Alternatively, we could use the full component of nine mesons, or

$$B \otimes \bar{B} \otimes B = 3 \otimes 3 \otimes 3 = 3 \oplus 3 \oplus 6 \oplus 15 \quad (3')$$

which does not help.

the  $\Lambda^0 (I = 0, I_3 = 0)$  (see Fig. 1).  $SU(3)$  symmetry requires that the interaction properties shall be invariant with respect to a rotation of these axes by  $120^\circ$ . Under such a rotation, the new vertical axis corresponds to the charge, while the horizontal axis represents the third component of another conserved vector, the  $U$ -spin (Fig. 2). Such a rotation, however, leads to a mixing of the  $\Sigma^0$  and  $\Lambda^0$ , such that the new combinations become  $Y^0 = \frac{1}{2}(-\Sigma^0 + \sqrt{3}\Lambda^0)$ , corresponding to  $U = 1$ ,  $U_3 = 0$ , and  $Z^0 = \frac{1}{2}(\sqrt{3}\Sigma^0 + \Lambda^0)$  for which  $U = 0$ ,  $U_3 = 0$ .

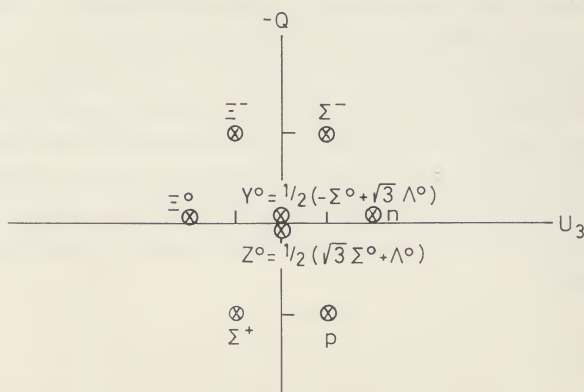


Fig. 2. The baryon octet in  $Q-U_3$  space, obtained by rotation of Fig. 1 through  $120^\circ$  degrees.

The symmetry-breaking interaction (which leads to the mass splitting), is charge-independent, i.e., a scalar in  $I$ -space. However, if we take it to have the properties of a vector in  $U$ -space, we may assume a Hamiltonian of the form

$$H = A + BU_3. \quad (4)$$

In this case, considering the  $U$ -spin triplet

$$M_{\Xi^0} = M_0 - b$$

$$M_{Y^0} = M_0 = \frac{1}{4}M_{\Sigma^0} + \frac{3}{4}M_{\Lambda^0}$$

(note that there is no off-diagonal  $\Sigma^0 - \Lambda^0$  matrix element, since  $I$ -spin is conserved by the symmetry-breaking interaction) and

$$M_n = M_0 + b. \quad (5)$$

Combining these, we obtain the famous relationship

$$\frac{M_N + M_{\Xi}}{2} = \frac{3M_{\Lambda} + M_{\Sigma}}{4} \quad (6)$$

which yields, for the known masses

$$1128.2 \stackrel{?}{=} 1134.8$$

which is excellent agreement.

In the case of the  $S = \frac{3}{2}^+$  decet,  $[N^*(I = \frac{3}{2}), Y_1^*(I = 1), \Xi^*(I = \frac{1}{2}), \Omega^-(I = 0)]$ , we may apply the same interaction (Eq. 4) to the negative  $U$ -spin quartet  $[N^{*-}(U_3 = \frac{3}{2}), Y_1^{*-}(U_3 = \frac{1}{2}), \Xi^{*-}(U_3 = -\frac{1}{2}), \Omega^-(U_3 = -\frac{3}{2})]$ , giving

$$\begin{aligned} M_{N^*} &= M_0 + \frac{3}{2}b \\ M_{Y_1^*} &= M_0 + \frac{1}{2}b \\ M_{\Xi^*} &= M_0 - \frac{1}{2}b \\ M_{\Omega^-} &= M_0 - \frac{3}{2}b \end{aligned} \quad (7)$$

or the observed equal-spacing rule.

These are, of course, special cases of the general rule of Gell-Mann and Okubo

$$M = M_0\{1 + aY + b[I(I+1) - Y^2/4]\}. \quad (8)$$

#### ELECTROMAGNETIC MASS SPLITTING

(1) On the doublet scheme, the universality of the pion interaction predicts for the electromagnetic mass splittings within the baryon multiplets

$$M_n - M_p = M_{\Xi^0} - M_{\Xi^-}. \quad (9)$$

Experimentally,  $M_n - M_p = 1.30$  MeV, while  $M_{\Xi^0} - M_{\Xi^-} = -6.5 \pm 1.0$  MeV. This discrepancy could hardly be removed by the (weaker)  $K$ -meson interactions.

(2) Since the predictions for the Fermi-Yang-Sakata model are dependent on the details of how the observed baryons are constructed, it is not especially fruitful to consider its predictions in this case.

(3) To obtain the electromagnetic splittings for the octet model, we note that the electromagnetic interactions are  $U$ -spin independent,

i.e., they are the same for all members of a given  $U$ -spin multiplet. Thus, we have for the baryons (Fig. 2)

$$\begin{aligned}\delta M_{\Xi^-} &= \delta M_{\Sigma^-} \\ \delta M_{\Xi^0} &= \delta M_n \\ \delta M_{\Sigma^+} &= \delta M_p\end{aligned}\tag{10}$$

or, combining these

$$(\delta M_{\Xi^-} - \delta M_{\Xi^0}) = (\delta M_{\Sigma^-} - \delta M_{\Sigma^+}) - (\delta M_n - \delta M_p).\tag{11}$$

The measured values yield

$$(6.5 \pm 1.0) \stackrel{?}{=} (7.7 \pm 0.3) - 1.3 = (6.4 \pm 0.3)$$

in almost too-good agreement.

Similar considerations can be applied to the electromagnetic mass splittings within the  $S = \frac{3}{2}^+$  decet.

#### MAGNETIC MOMENTS

(1) On the doublet model, since the pion interactions are identical, the moments, neglecting the symmetry-breaking effects, are

$$\begin{aligned}\mu_p &= \mu_{\Sigma^+} = -\mu_{\Sigma^-} = -\mu_{\Xi^-} \\ \mu_n &= \mu_{Y^0} = -\mu_{Z^0} = -\mu_{\Xi^0}.\end{aligned}\tag{12}$$

The relative signs are obvious from a consideration of the forms of the charge-independent pionic Yukawa reactions, e.g.,

$$\begin{aligned}p &\rightleftharpoons -\sqrt{\frac{1}{3}}p\pi^0 + \sqrt{\frac{2}{3}}n\pi^+ \\ \Xi^- &\rightleftharpoons \sqrt{\frac{1}{3}}\Xi^-\pi^0 - \sqrt{\frac{2}{3}}\Xi^0\pi^-.\end{aligned}\tag{13}$$

Recalling the definitions of the  $Y^0$  and  $Z^0$  particles in terms of  $A^0$  and  $\Sigma^0$  (section 1 of the Introduction) and noting that the magnetic moment operator does not mix doublets

$$\langle Y^0 | \mu | Z^0 \rangle = 0\tag{14}$$

one easily obtains

$$\mu_{A^0} = \mu_{\Sigma^0} = \frac{1}{2}(\mu_{Y^0} + \mu_{Z^0}) = 0\tag{15}$$

$$\mu_{A^0 - \Sigma^0} = \langle A^0 | \mu | \Sigma^0 \rangle = -\frac{1}{2}(\mu_{Y^0} - \mu_{Z^0}) = -\mu_n.\tag{16}$$



(2) The moments obtained on the basis of the Fermi-Yang-Sakata model are, again, highly model dependent. However, the symmetry imposed by SU(3) leads to a set of relationships among which the Sakata model may be considered as a special case.

(3) There are a number of equivalent ways of deriving the relationships among the magnetic moments according to SU(3). The most straightforward is to take advantage of the  $U$ -spin independence of the electromagnetic interactions, i.e., that the moments must be the same for all the members of a  $U$ -spin multiplet. Hence (see Fig. 2)

$$\begin{aligned}\mu_{\Sigma^+} &= \mu_p \\ \mu_{\Sigma^-} &= \mu_{\Xi^-} \\ \mu_n &= \mu_{\Xi^0} = \frac{1}{4}\mu_{\Sigma^0} + \frac{3}{4}\mu_{\Lambda^0} - \frac{1}{2}\sqrt{3}\mu_{\Lambda^0 - \Sigma^0}.\end{aligned}\quad (17)$$

Confining ourselves, for the moment, to the relations among the neutral baryons, we may obtain another equation from, among a number of possibilities \*, the condition that the matrix element of the magnetic moment operator vanishes between the  $U_3 = 0$  members of the  $U$ -spin triplet and singlet

$$\langle Y^0 = \frac{1}{2}(-\Sigma^0 + \sqrt{3}\Lambda^0) | \mu | Z^0 = \frac{1}{2}(\sqrt{3}\Sigma^0 + \Lambda^0) \rangle = 0 \quad (18)$$

which yields

$$-\frac{1}{4}\sqrt{3}\mu_{\Sigma^0} + \frac{1}{4}\sqrt{3}\mu_{\Lambda^0} + \frac{1}{2}\mu_{\Lambda^0 - \Sigma^0} = 0. \quad (19)$$

Eqs. (19) and (17) can be rearranged to give

$$\frac{\mu_{\Sigma^0}}{\mu_n} = \frac{3\mu_{\Lambda^0}}{\mu_n} - 2 \quad (20)$$

$$\sqrt{\frac{1}{3}} \frac{\mu_{\Lambda^0 - \Sigma^0}}{\mu_n} = \frac{\mu_{\Lambda^0}}{\mu_n} - 1 \quad (21)$$

which are plotted in Figure 3.

\* Other, equivalent conditions leading to Eq. 19 are: behavior of the electromagnetic interaction as a vector in  $V$ -space (the space obtained by rotating the axes of figure 2 by 120 degrees); adoption of an analogous Gell-Mann, Okubo formula for the moments

$$\mu = A + BQ + C[U(U+1) - \frac{1}{4}Q^2].$$

This is as far as we can go with SU(3) without some further assumption relating to the origin of the baryon symmetries. Thus, for example, we might adopt the Sakata hypothesis that the electromagnetic interactions are invariant with respect to the operation  $n \leftrightarrow \Lambda^0$ . In this case

$$\mu_{\Lambda^0} = \mu_n \quad (22a)$$

leading to

$$\begin{aligned} \mu_{\Sigma^0} &= \mu_n \\ \mu_{\Lambda^0 - \Sigma^0} &= 0. \end{aligned} \quad (23a)$$

However, we know that the Sakata triplet does not give rise to the observed baryon multiplet assignments.

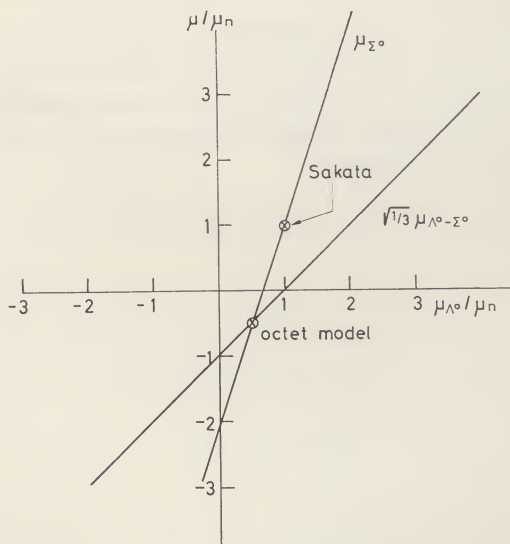


Fig. 3. Relations among the magnetic moments of the baryons according to SU(3). The solutions for the Sakata model and for the quark model are indicated.

Alternatively, we can start with a triplet of quarks  $B' = (p', n', \Lambda')$  of fractional charge  $(\frac{2}{3}e, -\frac{1}{3}e, -\frac{1}{3}e)$  and fractional hypercharge  $(Y = \frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$  and construct the baryons out of the combinations

$$B' \otimes B' \otimes B' = 1 \oplus 8 \oplus 8 \oplus 10. \quad (24)$$

In this case, the condition of invariance with respect to  $n' \leftrightarrow \Lambda'$  leads to \*

$$\mu_{\Lambda^0} = -\mu_{\Sigma^0} \quad (22b)$$

from which follows, from Eqs. (20) and (21),

$$\begin{aligned} \mu_{\Lambda^0} &= \frac{1}{2}\mu_n = -\mu_{\Sigma^0} \\ \mu_{\Lambda^0-\Sigma^0} &= -\frac{1}{2}\sqrt{3}\mu_n. \end{aligned} \quad (23b)$$

Recent measurements of the  $\Lambda^0$ -moment (Hill, Kycia, *et al.*) have confirmed the first of these predictions.

We may now return to Eqs. (17) for the predictions of SU(3) for the other baryons.  $\mu_{\Sigma^+}$  and  $\mu_{\Xi^0}$  are immediately given in terms of the known nucleon moments. One additional relationship serves to determine the rest. This may be obtained from the observation that the electromagnetic interactions behave as a vector in  $I$ -spin space (i.e.,  $\mu \propto I_3$  within a given  $I$ -spin multiplet). Hence

$$\mu_{\Sigma^0} = \frac{1}{2}(\mu_{\Sigma^+} + \mu_{\Sigma^-}) \quad (25)$$

which, together with Eqs. (17) and (23b) gives

$$\mu_{\Sigma^-} = \mu_{\Xi^-} = -(\mu_p + \mu_n). \quad (26)$$

Thus, by straightforward application of the symmetry requirements of the SU(3)-quark model, one can obtain the moments of all the baryons in terms of those of the nucleons. It is now well known that this set of relationships becomes complete with the further prediction of the SU(6)-quark model of

$$\frac{\mu_p}{\mu_n} = -\frac{3}{2}. \quad (27)$$

\* Specifically, the vanishing of the off-diagonal matrix elements for  $U$ -spin eigenstates defined by this operation.

# SOME THEORETICAL CONSIDERATIONS ON THE REAL PART OF THE FORWARD SCATTERING AMPLITUDE

TOICHIRO KINOSHITA \*

*Cornell University, Ithaca, New York*

and

N. N. KHURI

*Rockefeller Institute, New York, New York*

*(Received May 3, 1965)*

Recently several groups of physicists have carried out very precise measurements of cross sections for proton-proton and pion-proton elastic scattering in the Coulomb interference region [1]. The observed results can be understood most easily if one assumes the existence of large negative real part for the forward scattering amplitude. At present this interpretation seems to be somewhat ambiguous. In particular, in the case of proton-proton scattering, possible spin dependence of the forward scattering amplitude must be explored carefully before one can draw firm conclusion on the real part. However, if this interpretation is essentially correct, it will enable us to study the structure of strongly interacting particles in greater details.

It has been pointed out already that the sign and magnitude of the observed real part may be explained if one assumes Regge poles of reasonable properties [2]. In the absence of relativistic theory of Regge poles, however, it is doubtful whether such an approach leads us to a better understanding of high energy physics beyond the level of phenomenology.

In another approach the real part of the scattering amplitude is calculated by means of the forward dispersion relation using the observed total cross section as input and assuming simple energy dependence of the total cross section at higher energies [3]. The result of calculation seems to agree reasonably well with the observation. In this approach, however, it will not be possible to find out whether or not the experi-

\* Supported in part by the U.S. Office of Naval Research.



mental results contradict the dispersion relation itself because the calculation depends on the assumed value of the total cross section at ultra-high energies which is guaranteed to be unobservable by any future accelerator. This situation may not be improved substantially until a dynamical theory of strong interaction is discovered which gives a definite prediction about the behavior of the scattering amplitude at finite as well as infinite energies.

In the absence of such a theory, it may not be pointless to ask whether or not the present dispersion relations can be used in such a manner that the influence of the asymptotic behavior of the scattering amplitude is eliminated or reduced as much as possible. In this note we should like to describe briefly some results of recent efforts initiated by this question. In the first half we shall discuss new type of sum rules satisfied by the real part of the forward scattering amplitude [4]. The method developed for this purpose was found to be useful for the study of the relation between the asymptotic behavior of the forward scattering amplitude and that of the ratio of the real and imaginary parts of the scattering amplitude [5]. This will be discussed in the second half.

In order to emphasize the usefulness of these sum rules, it will be appropriate to call attention to an important development in the field of axiomatic field theory. Recently Hepp [6] has shown that the Lehmann-Symanzik-Zimmermann formalism of quantum field theory can be rigorously derived from the Wightman axioms, and also that the forward dispersion relation (for pion-nucleon scattering) is valid and requires only finite number of subtractions in Wightman theory. This is the first time that the finiteness of the subtraction procedure was actually proved. With the result of Hepp and the new sum rules at hand, we are now in the position of making an experimental test of some consequences of axiomatic field theory. No longer axiomatic field theory is as far removed from the physical world as it once seemed to be. If any disagreement were uncovered between the experimental data and the sum rules, we would be forced to reexamine some of the foundations of local field theory.

For the sake of concreteness we shall limit ourselves to pion nucleon scattering. We denote by  $E$  the total energy of the incident pion in the laboratory system, and by  $f_{\pm}(E)$  the forward scattering amplitude

for  $\pi^\pm p$  scattering, respectively. We shall be concerned exclusively with the symmetric amplitude  $f(E)$  defined as follows:

$$f(E) = \frac{1}{2}[f_+(E) + f_-(E)] - \text{nucleon pole terms.} \quad (1)$$

As is well known from axiomatic field theory,  $f(E)$  has the properties:

- i)  $f(E)$  is analytic in  $E$  and regular in the cut  $E$  plane with cuts running from  $-\infty$  to  $-\mu$  and from  $\mu$  to  $+\infty$ ,
- ii)  $f(E+i0) = f^*(E-i0)$ ,
- iii)  $f(E+i0) = f(-E-i0)$ ,
- iv) unitarity requires that the discontinuity  $\text{Im} f(E+i0)$  on the cut  $E \geq \mu$  is positive.

In general the discontinuity  $\text{Im} f(E+i0)$  is a tempered distribution. Thus it is necessary to regularize it over a small interval of values of  $E$ . We shall assume that this averaging is already done and  $\text{Im} f(E+i0)$  is continuous on the real  $E$  axis.

If we denote by  $f(E, \cos \theta)$  the scattering amplitude as a function of energy  $E$  and the center-of-mass scattering angle  $\theta$ , it is subject to the inequality

$$\text{v) } |f(E, \cos \theta)| < C|E|^N, \quad \text{for } E \rightarrow \infty,$$

for any  $\cos \theta$  inside the Lehmann ellipse, as was shown by Hepp [6]. From this and the unitarity condition it follows that

$$|f(E)| < C|E|^2(\ln |E|)^2 \quad (2)$$

as  $|E| \rightarrow \infty$  in all direction in the  $E$  plane. This property was derived for real  $E$  by Greenberg and Low [7]. It is generalized to the case  $|E| \rightarrow \infty$  making use of the Phragmén-Lindelöf theorem [8].

The properties i) . . . v) and (2) are enough to insure the validity of the dispersion relation for  $f(E)$  with at most three subtractions. However, if one adds to these conditions the physical requirement that  $\text{Im} f/\text{Re} f$  does not tend to zero as  $E \rightarrow +\infty$ , (2) can be replaced by the stronger inequality

$$|f(E)| < C|E|^{2-\varepsilon}, \quad \varepsilon > 0, \quad (3)$$

for  $|E| \rightarrow \infty$  [9]. We wish to stress that the requirement  $\text{Im} f/\text{Re} f \not\rightarrow 0$  as  $E \rightarrow \infty$  has not yet been proved to be a consequence of axiomatic field theory. Nevertheless, it seems to be a reasonable feature of theory which has an infinite number of open inelastic channels as  $E \rightarrow \infty$ .

From the assumptions i) . . . v) and (3) it follows that  $f(E)$  satisfies

the twice subtracted dispersion relation

$$f(E) - f(0) = \frac{2E^2}{\pi} \int_{\mu}^{\infty} dE' \frac{\text{Im} f(E')}{E'(E'^2 - E^2)}. \quad (4)$$

This relation, or more exactly the analytic properties it implies, can, at least in principle, be tested experimentally. In practice, however, the relation (4) has two disadvantages. The first is the fact that it involves integrations up to infinite energies. The second is that, as  $\text{Im } E \rightarrow 0$ , principal value integrals have to be used in (4). Both disadvantages can be avoided by converting the dispersion relation (4) into sum rules for  $\text{Re} f(E)$ . Although it is a direct consequence of (4), it gives for practical purposes a better tool for testing the consequences of local field theory. These sum rules also show explicitly the fact that a large and repulsive real part at high energies, if maintained for a certain large energy range, will lead to a contradiction with (4).

To derive the sum rules, we shall consider the function

$$g(E) = \int_0^E \frac{f(E') - f(0)}{E'^2} dE', \quad \text{Im } E \geq 0, \quad (5)$$

where the integration path should lie entirely in the upper half  $E$  plane. Now, dividing both sides of (4) by  $E^2$ , interchanging the order of integration, and integrating from 0 to  $E$  along the radial direction, we obtain, after taking the real parts,

$$\text{Re } g(E) = \frac{1}{\pi} \int_{\mu}^{\infty} dE' \frac{\text{Im} f(E')}{E'^2} \ln \left| \frac{E' + E}{E' - E} \right|, \quad 0 < \arg E < \pi. \quad (6)$$

We note that  $\ln |(E' + E)/(E' - E)| \geq 0$  for  $0 \leq \arg E \leq \pi/2$ . Since  $\text{Im} f(E') > 0$  for  $E' > \mu$ , we see that  $\text{Re } g(E) > 0$  for all  $E$  such that  $0 \leq \arg E < \pi/2$ . In particular, for positive real  $E$  we obtain

$$\int_0^E \frac{\text{Re} f(E') - f(0)}{E'^2} dE' = \frac{1}{\pi} \int_{\mu}^{\infty} dE' \frac{\text{Im} f(E')}{E'^2} \ln \left| \frac{E' + E}{E' - E} \right|. \quad (7)$$

The integrand on the right-hand side of (7) is always positive. If the integration is cut off at the maximum energy,  $E_m$ , for which one has data on the total cross section, one obtains an inequality which should be satisfied regardless of the actual value of the total cross section at

super-high energies:

$$\int_0^E \frac{\operatorname{Re} f(E') - f(0)}{E'^2} dE' > \frac{1}{\pi} \int_{\mu}^{E_m} dE' \frac{\operatorname{Im} f(E')}{E'^2} \ln \left| \frac{E' + E}{E' - E} \right|. \quad (8)$$

It is evident from (7) why a large and negative  $\operatorname{Re} f$  is dangerous to analyticity. The present data give, roughly,  $\operatorname{Re} f \sim -cE$  where  $c$  is about  $1/20\pi$  of the total cross section,  $(\sigma_+ + \sigma_-)/2 = (4\pi/k) \operatorname{Im} f$ . Clearly such a behavior, if maintained to higher energies, will not only make the left-hand side of (7) smaller, but might even make it negative for large enough  $E$ .

In an actual comparison of (7) or (8) with the data, one has to know  $\operatorname{Re} f$  in the unphysical region  $0 \leq E < \mu$ . This can be obtained from the dispersion relation. It is well known that the dispersion relation is reliable for low energies. As an alternative we may use this information and subtract all the low energy data from (7). For example, if the dispersion relation is known to be approximately valid for  $E \leq E_1$  ( $E_1 \approx 1-3$  GeV), then a relation like (7) holds with  $E = E_1$ . Subtracting this relation from (7), we obtain

$$\int_{E_1}^E \frac{\operatorname{Re} f(E') - f(0)}{E'^2} dE' = \frac{1}{\pi} \int_{\mu}^{\infty} dE' \frac{\operatorname{Im} f(E')}{E'^2} \ln \left| \frac{(E' + E)(E' - E_1)}{(E' - E)(E' + E_1)} \right|, \quad (9)$$

where  $E > E_1$ . The integration on the left-hand side now involves only the high energy domain. For  $E' > E$  the integrand on the right-hand side is positive. We can therefore cut off the integration on the right at some  $E_m \geq E > E_1$ , and obtain a lower bound for the integral on the left. Thus we obtain the sum rule

$$\int_{E_1}^E \frac{\operatorname{Re} f(E') - f(0)}{E'^2} dE' > \frac{1}{\pi} \int_{\mu}^{E_m} dE' \frac{\operatorname{Im} f(E')}{E'^2} \ln \left| \frac{(E' + E)(E' - E_1)}{(E' - E)(E' + E_1)} \right|. \quad (10)$$

The only quantity in this inequality which is not obtainable immediately from the data is  $f(0)$ . But for estimating it one can always use the dispersion relation.

The present data are still sketchy for  $\operatorname{Re} f$ . But just to see how serious the situation is, take  $E_1 = 4$  GeV,  $E_m = E = 30$  GeV. Furthermore



assume that

$$\operatorname{Im} f(E') = cE', \quad \operatorname{Re} f(E') = f(0) + \alpha cE' \quad (11)$$

for  $4 \text{ GeV} \leq E' \leq 30 \text{ GeV}$ . If the expression (11) is substituted in (10), it is seen that the inequality (10) will be violated if  $\alpha < -\frac{1}{3}$ .

The sum rule (10) still depends on the unphysical quantity  $f(0)$ . This may be avoided by carrying out the subtraction at the threshold  $E = \mu$  instead of  $E = 0$ . In fact, following the same procedure as above, Martin [10] obtained the inequality

$$\begin{aligned} \int_{\mu}^E \frac{\operatorname{Re} f(E') - f(\mu)}{(E'^2 - \mu^2)^{\frac{3}{2}}} E' dE' > \\ > \frac{1}{\pi} \int_{\mu}^{E_m} \frac{\operatorname{Im} f(E') E' dE'}{(E'^2 - \mu^2)^{\frac{3}{2}}} \ln \left| \frac{(E'^2 - \mu^2)^{\frac{1}{2}} + (E^2 - \mu^2)^{\frac{1}{2}}}{(E'^2 - \mu^2)^{\frac{1}{2}} - (E^2 - \mu^2)^{\frac{1}{2}}} \right|, \quad (12) \end{aligned}$$

which is free from unphysical quantities. It should be noted, however, that the accuracy of this inequality still depends on the determination of nucleon pole terms which must be added to  $\operatorname{Re} f$  to obtain the real part of the *actual* scattering amplitude. But it seems that it is not easy to go much further in this direction.

We should like to devote the rest of this note to the study of the asymptotic behavior of the forward scattering amplitude making use of some remarkable property of the function  $g(E)$  [5]. This is that  $g(E)$  does not take any value more than once in the upper half  $E$  plane. We shall first demonstrate this property which is called univalence. For this purpose let us note that the function

$$h(E) = \frac{f(E) - f(0)}{E} \quad (13)$$

has the following properties: a)  $h(E)$  is regular for  $\operatorname{Im} E > 0$  and continuous for  $\operatorname{Im} E \geq 0$ , b)  $\operatorname{Im} h(E) > 0$  when  $\operatorname{Im} E > 0$  (thus  $h(E)$  is the so called Herglotz function), c)  $h(i\lambda)$ ,  $\lambda$  real and positive, is purely imaginary, d)  $\operatorname{Re} h(E + i0) = -\operatorname{Re} h(-E + i0)$  and  $\operatorname{Im} h(E + i0) = \operatorname{Im} h(-E + i0)$  for real  $E$ . Thus, if we consider the mapping of the upper half  $E$  plane by the function  $h(E)$ , the image will lie in the upper half  $h$  plane. If this mapping were not only regular but also univalent, we could apply powerful theorems of geometric function theory to study its properties.

In general there is no guarantee from field theory that the forward scattering amplitude is a regular univalent function of energy variable in the upper half  $E$  plane. However such univalent functions can be easily constructed from the scattering amplitude. One such function is  $g(E)$  defined by (5) which, in terms of  $h(E)$ , can be written as

$$g(E) = \int_0^E \frac{h(E')}{E'} dE', \quad \text{Im } E \geq 0, \quad (14)$$

where the path of integration lies entirely in the upper half  $E$  plane.

One can easily check that  $g(E)$  has the following properties: 1)  $g(E)$  is regular in  $\text{Im } E > 0$  and continuous in  $\text{Im } E \geq 0$ , 2)  $\text{Im } g(E) > 0$  if  $\text{Im } E > 0$ , 3)  $g'(E) \neq 0$  everywhere in  $\text{Im } E > 0$ , 4)  $\text{Im } g(-E+i0) = \text{Im } g(E+i0)$  and  $\text{Re } g(-E+i0) = -\text{Re } g(E+i0)$  for all real  $E$ , 5) for  $E > \mu$ ,  $\text{Im } g(E+i0)$  is nonnegative and increases monotonically along the positive real  $E$  axis, 6)  $\text{Re } g(E+i0)$  is nonnegative and increases monotonically in the interval  $0 \leq E \leq \mu$ , and finally 7)  $g(i\lambda)$  for real positive  $\lambda$  is purely imaginary and its magnitude increases monotonically as  $\lambda$  increases.

As is seen from the property 2),  $g(E)$  maps the upper half  $E$  plane into a domain  $G$  located in the upper half  $g$  plane. We know from 3) that this mapping is locally one-to-one everywhere in the upper half  $E$  plane. The mapping will be globally univalent if the boundary curve of  $G$  does not have double points [11]. Let us denote by  $\Gamma_1$  and  $\Gamma_2$  the images of the negative and positive real  $E$  axis respectively. We know from 6) that the part of  $\Gamma_2$  corresponding to  $0 \leq E \leq \mu$  does not intersect with itself and lies on the positive real  $g$  axis. For  $E > \mu$ ,  $g(E)$  becomes complex and the corresponding part of  $\Gamma_2$  goes away monotonically from the real  $g$  axis according to 5). Thus  $\Gamma_2$  cannot have double point. The same holds for  $\Gamma_1$ . Hence the only remaining possibility is that  $\Gamma_1$  and  $\Gamma_2$  have some common points. Because of the monotonicity and symmetry of  $\Gamma_1$  and  $\Gamma_2$  such a common point could be found only on the imaginary  $g$  axis. It is easily seen that this can happen only if this common point corresponds to  $E = \infty$ . Otherwise neither  $\Gamma_1$  nor  $\Gamma_2$  touches or crosses the imaginary  $g$  axis. Thus the boundary curve of  $G$  has no double point, which proves the univalence of  $g(E)$  in the upper half  $E$  plane.

Once we know that  $g(E)$  is univalent, we can make use of various

theorems in geometric function theory. For instance, applying Koebe's theorem [12], we obtain [5]

$$\int_0^{E_\lambda} \frac{\operatorname{Re} f(E') - f(0)}{E'^2} dE' \geq \frac{1}{\lambda} |f(i\lambda) - f(0)| \quad (15)$$

for any positive real  $\lambda$ , where  $E_\lambda$  and  $\lambda$  are related by

$$\int_\mu^{E_\lambda} \frac{\operatorname{Im} f(E')}{E'^2} dE' = |g(i\lambda)|. \quad (16)$$

From the dispersion relation (4) one can get a lower bound for the right-hand side of (15) which is independent of the value of the total cross section for  $E' > E_m$ . Namely one can write

$$\frac{|f(i\lambda) - f(0)|}{\lambda} > \frac{2\lambda}{\pi} \int_\mu^{E_m} \frac{\operatorname{Im} f(E')}{E'(E'^2 + \lambda^2)} dE'. \quad (17)$$

Although the sum rule (15) (together with (17)) is not as good as (8), it has an advantage over (8) in that it is much less sensitive to the value of the total cross section for large  $E'$ .

Koebe's theorem may also be used to examine the asymptotic behavior of  $g(E)$  for very large  $E$ . However the best results are obtained if we make use of Ahlfors' distortion theorem [13]. We shall mention some of the results without proof [14].

As was mentioned already,  $\operatorname{Re} g(E)$  never becomes negative for real positive  $E$ . Thus  $\operatorname{Re} g(E)/\operatorname{Im} g(E)$  is also nonnegative for real positive  $E$ . Various cases can be considered depending on the asymptotic value of this ratio.

A. Suppose that we can find positive constants  $\alpha$  and  $E_0$  such that

$$\frac{\operatorname{Re} g(E)}{\operatorname{Im} g(E)} \geq \tan \pi\alpha, \quad 0 < \alpha < \frac{1}{2} \quad (18)$$

for all real positive  $E$  greater than  $E_0$ . Then  $g(E)$  has the lower bound

$$|g(E)| \geq C \left( \frac{E}{E_0} \right)^{2\alpha} \quad (19)$$

for all  $E > E_1$  where  $E_1$  is some constant greater than  $E_0$ .

B. If we can find a positive constant  $\alpha'$  such that

$$\frac{\operatorname{Re} g(E)}{\operatorname{Im} g(E)} \leq \tan \pi \alpha', \quad 0 < \alpha' < \frac{1}{2} \quad (20)$$

for all  $E > E_0$ ,  $g(E)$  has the upper bound

$$|g(E)| \leq C' \left( \frac{E}{E_0} \right)^{2\alpha'} \quad (21)$$

for all  $E > E_1$ .

If  $\operatorname{Re} g(E)$  is bounded or grows much less rapidly than  $\operatorname{Im} g(E)$  as  $E \rightarrow +\infty$ , the condition (18) may not be convenient since we cannot find any positive  $\alpha$ . In such a case we may characterize in (18) the asymptotic behavior of  $\operatorname{Re} g(E)/\operatorname{Im} g(E)$  by a function  $\alpha(E)$  which decreases monotonically to zero as  $E \rightarrow +\infty$ . In this manner we obtain

C. If  $g(E)$  satisfies

$$\frac{\operatorname{Re} g(E)}{\operatorname{Im} g(E)} \leq \frac{C}{(\ln E)^a}, \quad 0 \leq a < 1 \quad (22)$$

for all  $E > E_0$ , we obtain

$$|g(E)| \leq C' (\ln E)^\gamma \quad (23)$$

for all  $E > E_1$ . Here  $\gamma$  is greater than any positive number.

D. If  $g(E)$  satisfies

$$\frac{\operatorname{Re} g(E)}{\operatorname{Im} g(E)} < \frac{C}{(\ln E)^a}, \quad a > 1 \quad (24)$$

for all  $E > E_0$ ,  $g(E)$  is bounded as  $E \rightarrow +\infty$  and thus  $\lim_{E \rightarrow +\infty} \operatorname{Re} g(E) = 0$ .

Another way to characterize the asymptotic behavior when  $\alpha = 0$  is to make an assumption on  $\operatorname{Re} g(E)$  itself rather than the ratio  $\operatorname{Re} g/\operatorname{Im} g$ . For instance:

E. If  $g(E)$  satisfies the condition

$$\operatorname{Re} g(E) \geq b \quad (25)$$



for all  $E > E_0$ ,  $g(E)$  has the lower bound

$$|g(E)| \geq \frac{2b}{\pi} \ln \left( \frac{E}{E_0} \right) + \text{constant} \quad (26)$$

for all  $E \gg E_0$ .

F. If  $g(E)$  satisfies

$$\text{Re } g(E) \leq b' \quad (27)$$

for all  $E > E_0$ , then for sufficiently large  $E$  we get

$$|g(E)| \leq \frac{2b'}{\pi} \ln \left( \frac{E}{E_0} \right) + \text{constant}. \quad (28)$$

The method used to obtain (26) and (28) can be easily generalized to the following cases:

G. Suppose we find  $\nu (> 1)$  such that

$$\frac{\text{Re } g(E)}{\nu(\text{Im } g(E))^{1-1/\nu}} \geq b, \quad E > E_0. \quad (29)$$

Then we obtain

$$\text{Im } g(E) \geq C \left( \ln \left( \frac{E}{E_0} \right) \right)^\nu \quad (30)$$

for all  $E \gg E_0$ .

H. If we can find  $\nu (> 1)$  such that

$$\frac{\text{Re } g(E)}{\nu(\text{Im } g(E))^{1-1/\nu}} \leq b', \quad E > E_0, \quad (31)$$

then for  $E \gg E_0$  we have

$$\text{Im } g(E) \leq C' \left( \ln \left( \frac{E}{E_0} \right) \right)^\nu. \quad (32)$$

To discuss physical implication of these results, we shall now assume that the scattering amplitude  $f(E)$  satisfies the Froissart bound

$$|f(E)| \leq C|E|(\ln |E|)^2 \quad (33)$$

for all energies  $E$  greater than some  $E_0$ . Then  $g(E)$  satisfies the bound

$$|g(E)| \leq C(\ln |E|)^3. \quad (34)$$

The first thing to notice is that, if (34) is valid, the theorems given

above impose severe restrictions on the possible asymptotic behavior of the ratio  $\text{Re } g/\text{Im } g$ . For example one sees from (19) and (23) that, if  $\text{Re } g(E)/\text{Im } g(E) \geq C(\ln E)^{-a}$ ,  $0 < a < 1$ , for all  $E > E_0$ , then  $g(E)$  grows more rapidly than the right-hand side of (34). Thus such an asymptotic behavior of  $\text{Re } g/\text{Im } g$  must be excluded if (34) is valid. On the other hand, if  $\text{Re } g(E)/\text{Im } g(E) \leq C(\ln E)^{-a}$ ,  $a > 1$ , for all  $E > E_0$ , then  $|g(E)|$  is bounded by a constant as  $E \rightarrow +\infty$ , as is seen from  $D$ . This would correspond to the case where the total cross section vanishes faster than  $1/\ln E$  as  $E \rightarrow +\infty$ . If we exclude this case which does not seem to be of much physical interest, we find that there must be at least a sequence of points  $\{E_i\}$ ,  $E_i \rightarrow +\infty$  as  $i \rightarrow \infty$ , such that

$$\frac{C}{(\ln E_i)^{1+\varepsilon}} \leq \frac{\text{Re } g(E_i)}{\text{Im } g(E_i)} \leq \frac{C'}{(\ln E_i)^{1-\varepsilon'}} \quad (35)$$

holds, where  $\varepsilon$  and  $\varepsilon'$  are arbitrarily small positive numbers. (The upper bound of (35) can be replaced by  $C''/\ln E_i$  by more careful consideration.) Since  $g(E)$  is an integral of the scattering amplitude  $f(E)$ , (35) will be satisfied for all  $E > E_0$  if  $f(E)$  satisfies some smoothness requirement. Under the same assumptions, it will then be shown that the ratio  $\text{Re } f(E)/\text{Im } f(E)$  of the scattering amplitude itself satisfies a relation similar to (35).

However the most interesting consequences of the theorems  $A \dots H$  are obtained when we make the physical assumption that  $\text{Re } f(E)$  has a definite sign beyond a certain large energy  $E_1$ . For example, if  $\text{Re } f(E) \leq 0$  for all real  $E \geq E_1$ ,  $\text{Re } g(E)$  is monotonically decreasing for all  $E \geq E_1$  and thus

$$\text{Re } g(E) \leq \text{Re } g(E_1), \quad E \geq E_1. \quad (36)$$

According to (28) we therefore have the upper bound

$$|g(E)| \leq \frac{2}{\pi} \text{Re } g(E_1) \ln E + \text{constant} \quad (37)$$

for all  $E \gg E_1$ . This means that the total cross section must be bounded by some constant for almost all  $E$  in the sense that

$$\int_{E_1}^E \frac{\sigma(E')}{E'} dE' \leq C \ln E \quad (38)$$

as  $E \rightarrow +\infty$ .

On the other hand, if  $\operatorname{Re} f(E) \geq 0$  for all  $E \geq E_1$ , then  $\operatorname{Re} g(E)$  is monotonically increasing and

$$\operatorname{Re} g(E) \geq \operatorname{Re} g(E_1), \quad E \geq E_1. \quad (39)$$

From (26) we thus have

$$|g(E)| \geq \frac{2}{\pi} \operatorname{Re} g(E_1) \ln E + \text{constant}. \quad (40)$$

Thus, in this case the total cross section cannot go to zero smoothly as  $E \rightarrow +\infty$ . Conversely, if the total cross section diverges in such a way that  $\operatorname{Im} g(E) > C(\ln E)^v$ ,  $v > 1$ , as  $E \rightarrow +\infty$ , it is impossible to find a finite constant  $C'$  such that  $\operatorname{Re} g(E) < C'$  for all large enough  $E$ , as is seen from (28). This means that  $\operatorname{Re} g(E)$  must tend to infinity. In such a case  $\operatorname{Re} f(E)$  cannot stay negative for all large  $E$ . We may therefore conclude that the energy independence of the observed total cross section at high energy is closely related to the negative sign of the real part of the forward scattering amplitude.

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- 14) See reference 5 for the proof of these theorems. Results of this nature were first obtained in reference 8 starting from Meiman's theorems (N. N. Meiman, *Zh. Eksperim. i Teor. Fiz.* **43** (1962) 2277 [English transl.: *Soviet Phys.-JETP* **16** (1963) 1609]). Similar result was also obtained by Y. S. Jin and S. W. MacDowell (reference 9) making use of the phase representation of the forward scattering amplitude.



# A SYSTEMATICS OF HADRONS IN SUBNUCLEAR PHYSICS

YOICHIRO NAMBU

*The Enrico Fermi Institute for Nuclear Studies  
and the Department of Physics, The University of Chicago, Chicago, Illinois*

*(Received May 3, 1965)*

1.

With the recognition that the  $SU(3)$  symmetry is the dominant feature of the strong interactions, the main concern of the elementary particle theory has naturally become directed at the understanding of the internal symmetry of particles at a deeper level. An immediate question that arises in this regard is whether there are fundamental objects (such as triplets or quartets) of which all the known baryons and mesons are composed. These fundamental objects would be to the baryons and mesons what the nucleons are to the nuclei, and the electrons and nuclei are to the atoms. If that was really the case, it would certainly precipitate a new revolution in our conceptual image of the world. At the moment we can only hope that the question will be answered within the next ten to twenty years when the 100 GeV to 1000 GeV range accelerators will have been realized.

Even now, the amusing and rather embarrassing success of the  $SU(6)$  theory [1] lends support to the existence of those fundamental objects. It is embarrassing because this is basically a non-relativistic and static theory, and we do not know exactly how this can cover the realm of high energy relativistic phenomena.

Putting aside those theoretical difficulties mainly associated with relativity, let us make the working hypothesis that there are fundamental objects which are heavy ( $\gg 1$  GeV), though not necessarily stable, and that inside each baryon or meson they are combined with a large binding energy, yet moving with non-relativistic velocities. Though this might look like a contradiction, at least it does not violate the uncertainty principle in non-relativistic quantum mechanics since the range of the binding forces ( $10^{-14} - 10^{-13}$  cm) is large compared

to the Compton wave lengths of those constituents, and the strength of the forces can be arbitrarily adjusted. In other words, we have a model very similar to the atomic nuclei except for large binding energies. Theoretical justification of such a hypothesis must await future investigation.

In a previous article [2], we have put forward such a model with the following characteristic features.

1) There exist two fundamental fermion triplets  $t_1$  and  $t_2$  with charge assignments  $(1, 0, 0)$  and  $(0, -1, -1)$  for their three members. The baryons have the structure  $\sim t_1 t_1 t_2$ , and the mesons  $\sim at_1 \bar{t}_1 + bt_2 \bar{t}_2$ .

2) To  $t_1$  and  $t_2$  are assigned "charm" charge  $C = +1$  and  $-2$  respectively. Thus the baryons and mesons (zero triality states) have  $C = 0$ . The primary binding forces acting on them are proportional to  $C$ . Let us imagine these forces to be mediated by a field ( $C$ -field). The resulting Coulomb-like energy though probably of finite range, then stabilizes the  $C = 0$  ("uncharmed") systems against the  $C \neq 0$  ("charmed") states, such as the triplets themselves.

3) The  $SU(6)$  symmetry can be brought in, with the Pauli principle taken into account, since the constituent particles are non-relativistic. In another paper, we also considered a three-triplet model, in which  $t_1, t_2$  and  $t_3$  have charge assignments  $(1, 0, 0)$ ,  $(1, 0, 0)$  and  $(0, -1, -1)$  respectively. This has the advantage that the baryon states (the 56-dimensional representation of  $SU(6)$ ) may be realized with s-state triplets as  $\sim t_1 t_2 t_3$ .

The reasoning that has gone into the above stability problem is similar to the one used in nuclear physics in deriving the semi-empirical formula of Weizsäcker. The purpose of the present paper is to put this idea into a more precise form, even though the outcome should still be called at best semi-quantitative.

## 2.

Let us first consider states composed of an arbitrary number of  $t_1$  and  $t_2$ , but without antiparticles  $\bar{t}_1$  and  $\bar{t}_2$ . Their masses are  $M_1$  and  $M_2$ , respectively, and the "charm" numbers 1 and  $-2$ , as was mentioned already. The pairwise interaction energy through the  $C$ -field will depend on the spatial configurations of the particles, but we will rep-

resent it, in the first approximation, by a constant  $V_c$ , as long as the size of the system is comparable with the range of the force. If the number of  $t_1$ 's and  $t_2$ 's are  $n_1$  and  $n_2$ , respectively, the total energy of the system is

$$\begin{aligned}
 E(n_1, n_2) &= M_1 n_1 + M_2 n_2 + \\
 &\quad + V_c \frac{1}{2} n_1 (n_1 - 1) + 4 V_c \frac{1}{2} n_2 (n_2 - 1) - 2 V_c n_1 n_2 \\
 &= M_1 n_1 + M_2 n_2 + \frac{1}{2} V_c (n_1 - 2n_2)^2 - \frac{1}{2} V_c (n_1 + 4n_2) \quad (1) \\
 &= (M_1 - \frac{1}{2} V_c) n_1 + (M_2 - 2V_c) n_2 + \frac{1}{2} V_c C^2, \\
 C &= n_1 - 2n_2.
 \end{aligned}$$

As expected, the leading quadratic term depends only on the total charm  $C$ . If  $V_c$  is sufficiently large, this will favor  $C = 0$  as the lowest states, which means  $n_1 = 2n_2$ . Restricting ourselves to  $C = 0$  states now, the remaining terms are linear in  $n_1$  and  $n_2$ , implying a saturation property. With  $n_1 = 2n_2$ , we have

$$E(2n_2, n_2) = (2M_1 + M_2 - 3V_c) n_2. \quad (2)$$

From the physical requirement that this increases with  $n_2$  and that the baryon ( $n_2 = 1$ ) be lighter than the triplets, we further need

$$M_1, M_2 > 2M_1 + M_2 - 3V_c > 0. \quad (3)$$

Thus the energy surface in the  $n_1 - n_2$  plane has a valley running along the line  $C = n_1 - 2n_2 = 0$ , and its level rises linearly with increasing coordinates. However, it will be further necessary to make sure that the  $C = 0$  states are actually lower than their neighbors even for small  $n$ 's. Namely

$$\begin{aligned}
 E(2n_2 \pm 1, n_2) &> E(2n_2, n_2), \\
 E(2n_2, n_2 \pm 1) &> E(2n_2, n_2).
 \end{aligned} \quad (4)$$

This gives two more conditions

$$V_c - M_1 > 0, \quad 4V_c - M_2 > 0. \quad (5)$$

Combining Eqs. (3) and (5), we obtain

$$3V_c - 2M_1 > M_2 - M_1 > 3(V_c - M_1) > 0. \quad (6)$$

The second triplet, therefore, must be heavier than the first, but not

too much heavier. This is because we have to maintain a balance between the energy due to rest masses and that due to interaction.

Eq. (1) may be expressed in terms of  $C$  and the baryon number  $B$  if we make an appropriate assignment:  $B = x$  for  $t_1$  and  $B = y$  for  $t_2$ . Since the baryon  $\sim t_1 t_1 t_2$  has  $B = 1$ , we require  $2x + y = 1$ . Possible choices given in ref. [2] are

$$\begin{aligned}(x, y) &= (\tfrac{1}{3}, \tfrac{1}{3}) \\ \text{or } (0, 1) \\ \text{or } (1, -1).\end{aligned}\tag{7}$$

The numbers  $n_1$  and  $n_2$  may be then expressed in terms of  $C$  and  $B$  as

$$\begin{aligned}n_1 &= 2B + yC \\ n_2 &= B - xC\end{aligned}\tag{8}$$

and thus

$$\begin{aligned}E(B, C) &= \tfrac{1}{2}V_c C^2 + (2M_1 + M_2 - 3V_c)B \\ &\quad + [(M_1 - \tfrac{1}{2}V_c) - (2M_1 + M_2 - 3V_c)x]C.\end{aligned}\tag{9}$$

At this point we should add a reservation that the linear terms in the above mass formula are not as meaningful as the leading quadratic terms since the effects depending on spatial configurations, such as those due to the finite range character of the  $C$ -field and the exchange energy, can be of the same order as the former.

### 3.

In order to consider the meson states, we will next bring in anti-particles as well in the picture. We make the basic assumption that a system consists of definite numbers of  $n_1, \bar{n}_1, n_2, \bar{n}_2$  of  $t_1, \bar{t}_1, t_2$  and  $\bar{t}_2$ . This means that *we regard pair creation and annihilation as forbidden processes*, which is consistent with our basic non-relativistic approach.

The formula corresponding to Eq. (1) becomes

$$\begin{aligned}E(n_1, \bar{n}_1, n_2, \bar{n}_2) &= (M_1 - \tfrac{1}{2}V_c)(n_1 + \bar{n}_1) + (M_2 - 2V_c)(n_2 + \bar{n}_2) + \\ &\quad + \tfrac{1}{2}V_c C^2,\end{aligned}\tag{10}$$

$$C = n_1 - \bar{n}_1 - 2(n_2 - \bar{n}_2).$$



The requirement that  $E > 0$  demands

$$M_1 - \frac{1}{2}V_c > 0, \quad M_2 - 2V_c > 0 \quad (11)$$

in contrast to Eq. (5), which was derived for the special case  $\bar{n}_1 = \bar{n}_2 = 0$ . We find, together with Eqs. (3) and (5),

$$M_1 > V_c - M_1 > M_2 - 2V_c > 0 \quad (12)$$

which replaces Eq. (6).

We will now relate the constants  $M_1$ ,  $M_2$  and  $V_c$  to the baryon ( $t_1 t_1 t_2$ ) and meson ( $t_1 \bar{t}_1$  and  $t_2 \bar{t}_2$ ) masses  $m$ ,  $\mu_1$  and  $\mu_2$ :

$$\begin{aligned} m &= 2M_1 + M_2 - 3V_c, \\ \mu_1 &= 2M_1 - V_c, \\ \mu_2 &= 2M_2 - 4V_c, \end{aligned} \quad (13)$$

from which we obtain an identity

$$2\mu_1 + \mu_2 = 2m. \quad (14)$$

Because of this, we cannot determine the three unknowns  $M_1$ ,  $M_2$ ,  $V_c$  uniquely. Instead, we can express Eq. (10) in terms of  $\mu_1$  and  $\mu_2$ :

$$E(n_1, \bar{n}_1, n_2, \bar{n}_2) = \frac{1}{2}\mu_1(n_1 + \bar{n}_1) + \frac{1}{2}\mu_2(n_2 + \bar{n}_2) + \frac{1}{2}V_c C^2. \quad (15)$$

Turning to the relation (14), we put  $m \sim 1.2$  GeV,  $\mu_1 \sim 600$  MeV  $= \frac{1}{2}m$  corresponding to the average baryon and meson masses, and predict a value

$$\mu_2 \sim m \sim 2\mu_1 \quad (16)$$

for the second meson. This is not an unreasonable value in view of the fact that a large number of unidentified meson resonances seem to exist in this energy range. Eq. (15) reduces then to the simple form

$$E(n_1, \bar{n}_1, n_2, \bar{n}_2) = \frac{1}{2}\mu_1[n_1 + \bar{n}_1 + 2(n_2 + \bar{n}_2)] + \frac{1}{2}V_c^2. \quad (17)$$

It is rather surprising that such a naive picture as ours can yield non-trivial and qualitatively reasonable results.

By way of a remark, we note from Eq. (13) that

$$\begin{aligned} M_1 &= \frac{1}{2}V_c + \frac{1}{2}\mu_1 \sim \frac{1}{2}V_c, \\ M_2 &= 2V_c + \frac{1}{2}\mu_2 \sim 2V_c \sim 4M_1 \end{aligned} \quad (18)$$

since  $V_c \gg \mu_1, \mu_2$  by assumption. Interestingly enough, the above relation admits the interpretation that the mass of each triplet is made up of a self-energy due to the  $C$ -field plus a small "bare mass"  $\frac{1}{2}\mu$ .

4.

We will now turn to the three-triplet model [3] proposed as an alternative to the two-triplet model. The three triplets  $t_1, t_2$  and  $t_3$  altogether contain nine fermions  $T_{i\alpha}$ ,  $i, \alpha = 1, 2, 3$ , where the index  $i$  distinguishes different triplets, and  $\alpha$  the different members of a triplet. Two different SU(3) operations, called SU(3)' and SU(3)'', are introduced, acting respectively on  $\alpha$  and  $i$ , and in these spaces  $T_{i\alpha}$  behave as a representation  $(3, 3^*)$ . The electric charge is assigned to each particle according to

$$Q = I_3 + \frac{1}{2}Y' + I_3' + \frac{1}{2}Y'' \quad (19)$$

which takes integral values. In fact both  $T_{1\alpha}$  and  $T_{2\alpha}$  have the assignment  $(1, 0, 0)$ , and  $T_{3\alpha}$  have  $(0, -1, -1)$ , exactly like  $t_1$  and  $t_2$  of the previous two-triplet model.

An important difference from the two-triplet case is that instead of the charm gauge group  $U(1)$ , we have the group SU(3)''. The charm gauge field  $C$  must then be replaced by an octet of gauge fields  $G_\mu$ ,  $\mu = 1, \dots, 8$ , coupled to the infinitesimal SU(3)'' generators (currents)  $\lambda_\mu''$  of the triplets, with a strength  $g$ . For a system containing altogether  $N$  particles, the exchange of such fields between a pair then results in an interaction energy

$$\begin{aligned} V_G &= +g^2 \sum_{n>m} \lambda_\mu''^{(n)} \cdot \lambda_\mu''^{(m)} = \frac{1}{2}g^2 \left[ \sum_{n=1}^N \lambda_\mu''^{(n)} \right] \left[ \sum_{m=1}^N \lambda_\mu''^{(m)} \right] - \frac{1}{2}g^2 \sum_{n=1}^N \lambda_\mu''^{(n)} \lambda_\mu''^{(n)} \\ &= \frac{1}{2}g^2 [C_2 - NC_{20}], \end{aligned} \quad (20)$$

where  $\lambda_\mu^{(n)}$  refers to the  $n$ -th particle,  $C_2$  is the quadratic Casimir operator of SU(3), and  $C_{20} = 4/3$  is its value for a triplet representation  $D(1, 0)$  or  $D(0, 1)$ . In general  $C_2$  is given by

$$C_2(l_1, l_2) = \frac{1}{3}(l_1^2 + l_1 l_2 + l_2^2) + (l_1 + l_2) \quad (21)$$

for a representation  $D(l_1, l_2)$ .

Note that the only dependence on the total number  $N$  of constituents appears in the second term of Eq. (17).

We add to  $V_G$  the rest masses ( $M$  = common mass), and obtain the total energy

$$E = (M - \frac{1}{2}C_{20}g^2)N + \frac{1}{2}g^2C_2. \quad (22)$$

Bound states are characterized by  $V_G < 0$ , and the low lying states by the smallest value of  $C_2$ , namely  $C_2 = 0$  for the singlet  $D(0, 0)$ . For the latter,  $E$  is simply proportional to the total number  $N$  of constituents, starting with the meson ( $N = 2$ )  $\sim t_1 \bar{t}_1 + t_2 \bar{t}_2 + t_3 \bar{t}_3$  and the baryon ( $N = 3$ )  $\sim t_1 t_2 t_3$  (antisymmetric combination). Their masses are thus related by

$$\mu = 2(M - \frac{1}{2}C_{20}g^2) = \frac{2}{3}m, \quad (23)$$

and Eq. (22) becomes

$$E = \frac{1}{2}\mu N + \frac{1}{2}g^2C_2. \quad (24)$$

These are to be compared with Eqs. (14) and (15). Because of the high symmetry among the three triplets, we have found only one set of mesons with  $N = 2$ . In any case, the energy is simply proportional to the total number of constituents as long as  $C_2 = 0$ , as if it were made up of non-interacting basic units of mass  $\frac{1}{2}\mu$ .

5.

Having disposed of the gross mass spectrum of many-triplet compound systems, we now turn our attention to the "fine structure" of low lying states, which in our view comprise all the mesons and baryon resonances known so far. In all probability, however, our crude qualitative arguments are not really satisfactory for discussing these details. We will therefore restrict ourselves to general remarks only.

Because of our basic assumptions about the superstrong interactions and the static behaviour of particles, the dynamics we have been dealing with so far does not depend on the spin and the SU(3) spin variables, therefore the system possesses the symmetry of superstrong interactions, the SU(6) symmetry of combined spin and SU(3) spin, and the symmetry of orbital angular momentum. The overall Pauli principle imposes constraints among these symmetries, and thereby single out certain SU(6) and orbital states for the lowest configuration with respect to the superstrong interaction. The general

classification of these states can be done as in the case of nuclear and atomic physics, but this will be beyond the scope of the present paper.

In the three-triplet model, however, the problem is relatively simple if we take only s-state triplets. The low lying three particle configuration is a  $SU(3)''$  singlet, so the baryon must go into a complete symmetric  $SU(6)$  representation 56. No other states are possible without changing the spatial configuration, but this will cause some change in the superstrong interaction. For the mesons, we obviously obtain  $36 = 35 + 1$   $SU(6)$  states which are degenerate. These results are in accordance with those of the original  $SU(6)$  theory, as well as its "relativistic" version.

We must next discuss the two additional effects which do exist and tend to upset the symmetries. One arises from the internal motion of particles, and the other from the presence of virtual mesons. Contrary to the prevalent view, we regard the mesons as perturbing forces rather than the decisive factors in the physics of hadrons. Since the strong interactions are then merely first forbidden processes, so to speak, the meson and baryon resonances are really bound states decaying via violation of superstrong interaction symmetry. Nevertheless, these secondary effects can affect, and may even decide, the "fine structure" of low lying states. Perhaps we may compare the situation to the electronic levels of an atom where the main spectrum is determined by the static Coulomb force, and both the fine structure and the photon emission processes are higher order effects. In this sense, we do not necessarily find a contradiction between the present approach and the conventional strong interaction theory as far as the low lying states are concerned.

The reason we consider the strong interaction as generally symmetry breaking is that the virtual exchange of 36 virtual mesons do not possess an  $SU(6)$  symmetric form. An ideal  $SU(6)$  symmetric interaction would involve the 35 generators  $\chi_\mu$  as in Eq. (20):

$$V \sim \pm g^2 \sum_{n>m} \chi_\mu^{(n)} \chi_\mu^{(m)} = \pm \frac{g^2}{8} \sum_{n>m} [\frac{2}{3} \sigma_i^{(n)} \sigma_i^{(m)} + \lambda_\alpha^{(n)} \lambda_\alpha^{(m)} + \sigma_i^{(n)} \sigma_i^{(m)} \lambda_\alpha^{(n)} \lambda_\alpha^{(m)}]. \quad (25)$$

Viewed as a static force, this requires an exchange of 35 scalar and axial vector mesons (opposite parity to the known meson multiplet!)



if the relative signs of the various terms are to be correctly maintained for both particle-particle and particle-antiparticle interactions. [For processes involving meson-baryon scattering, however, Capps [4] and Belinfante and Cutkosky [5] have shown their compatibility with SU(6).]

Next consider the effect of the internal motion. This disturbs the basic symmetry in two senses. It mixes the Dirac spinor components, introducing corrections to the static superstrong forces. Further it simply adds the kinetic energy of orbital motion to the system. As far as the symmetry is concerned, these perturbations act like adding a neutral singlet meson with a suitable spin-parity. Its order of magnitude will depend on the internal velocity  $v$  of the particles, which should be of the order  $1/MR$  where  $R$  is the size of the system. If we take this correction to be of the order  $Mv^2 \sim 100$  MeV, and  $R \sim 1/M$ , we obtain the estimation

$$M \sim 10m, \quad \frac{v}{c} \sim \frac{1}{10}$$

as we did before [2].

6.

Finally we would like to comment on some obvious difficulties and intriguing problems concerning our model of the subnuclear structure of hadrons.

a) What is the origin of superstrong interactions?

Are these another kind of vector fields or something entirely new? If they are ordinary fields, their range must be at least of the order of the baryon size, and moreover sufficiently smooth and well-behaved in order to keep the kinetic energy small. It is conceivable that no single or a relatively few well defined meson states are responsible for this. A direct confirmation of such interactions would be difficult.

b) The magnetic moments of baryons, for example, agree closely with the SU(6) symmetry, yet obviously the bulk of contributions come from the meson cloud. This means that regardless of whether the meson cloud obeys SU(6) symmetry or not, the baryon should not be considered as composed of three bare triplets without structure. How, then, can we justify our picture that each system, including the mesons, is composed

of a definite number of triplets? The answer to this probably should be that the quantities like charm are at any instant well localized at a definite number of centers in space, and these centers are accompanied by large concentrations of energy, moving with slow velocities; whereas the quantities like ordinary charge are more uniformly spread out and carried by faster moving matter. In order to test such a picture experimentally, we would have to use some phenomena which depend on the energy distribution, the correlation functions of charges and energies at different points, the internal velocity of particles, etc.

c) The notion that decays and resonances are actually forbidden processes was first recognized as a surprising paradox in the process of adapting  $SU(6)$  to relativity. In our view, this is not only natural, but also simplifies the whole picture. We should be able to discuss the classes of first forbidden, second forbidden, etc. transitions, and they will be accessible to experimental test [6]. For this we should look especially for small, inconspicuous bumps in cross sections, many particle decay modes, and relatively rare events.

d) It has been widely speculated that an axial vector current conservation as relativistic chiral symmetry has physical significance. If this is actually the case, it is probably beyond the capacity of our extreme static approach, since we have first to explain away the large masses of triplets, even though we can formally apply group theoretical arguments and the Goldberger-Treiman type relations to individual problems.

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# A LORENTZ COVARIANT SUPERMULTIPLY SCHEME FOR STRONG INTERACTIONS

REINHARD OEHME

*The Enrico Fermi Institute for Nuclear Studies  
and the Department of Physics, The University of Chicago, Chicago, Illinois*

*(Received May 3, 1965)*

## 1. INTRODUCTION

We would like to describe in this article a Lorentz-covariant scheme for elementary particle interactions, which reproduces the successful features of non-relativistic  $SU(6)$ -models [1], and gives further interesting results. We know that it is apparently not possible to have a reasonable theory with finite supermultiplets which is Lorentz-invariant, and which also complies with all the basic assumptions of field theory or of dispersion theory [2]. Therefore, we aim only at an approximate scheme for  $S$ -matrix elements and form factors.

As a starting point, we assume that the fields describing the asymptotic, noninteracting particles are tensors of  $U_{\mathcal{G}}(12)[U(6, 6)]$  [3–5]. However, we require these tensors to satisfy the Bargmann-Wigner equations [6], which are not covariant with respect to  $U_{\mathcal{G}}(12)$ , but which define the physical particles in just such a way that we have the supermultiplet structure of  $U(6)$ . Several authors [5] have used the modified  $U_{\mathcal{G}}(12)$ -tensors in order to construct vertex-parts and four-particle amplitudes which are formally invariant under  $U_{\mathcal{G}}(12)$ , except for the intrinsic symmetry breaking due to the equations of motion. This scheme turns out to be too restrictive [7, 8].

We have proposed, therefore, a more general scheme [9, 10] where the  $U_{\mathcal{G}}(12)$ -invariance of a given amplitude is broken, not only by the Bargmann-Wigner equations, but also by the insertion of momentum spurions

$$S = (\Gamma_{\mu} \gamma_{\mu}) \otimes 1 \quad (1)$$

in arbitrary order. In Eq. (1), the vector  $\Gamma_{\mu}$  should be constructed out

of the linear independent momentum vectors available in the amplitude into which the spurion  $S$  is being inserted.

In general, momentum dependent terms in the Lagrangian will break  $U_{\mathcal{L}}(12)$ -symmetry down to  $U(3)$ . However, in situations where the directions of particle motions are restricted, the possibilities for symmetry-breaking by the momentum-dependent terms are also limited, and we are left with invariance of the corresponding amplitude with respect to certain compact subgroups of  $U_{\mathcal{L}}(12)$ .

It is easy to see that the spurion  $S$  does not give rise to any mass splitting if it is inserted into invariants like [9–11]

$$\bar{\Psi}_{ABC}(p)\Psi^{ABC}(p) \quad \text{and} \quad \bar{\Phi}_B^A(k)\Phi_A^B(k), \quad (2)$$

which may be considered as the mass terms for baryons and mesons, respectively. Here the tensors  $\Psi$  and  $\Phi$  are given by

$$\begin{aligned} \Psi^{ABC}(p) = & \frac{1}{m} \sqrt{\frac{3}{2}} \{ [(-i\gamma \cdot p + m)\gamma_\mu C]^{\alpha\beta} B_\mu^{\gamma,abc}(p) + \\ & + \frac{1}{3} [ [(-i\gamma \cdot p + m)\gamma_5 C]^{\alpha\beta} \varepsilon^{ab\zeta} B_\zeta^{\gamma,c}(p) + \text{cyclic} ] \}, \end{aligned} \quad (3)$$

and

$$\Phi_B^A(k) = \left\{ \left[ \left( 1 - \frac{i\gamma \cdot k}{\mu} \right) \gamma_5 \right]_\beta^\alpha P_b^a(k) + \left[ \left( 1 - \frac{i\gamma \cdot k}{\mu} \right) (\gamma \cdot e) \right]_\beta^\alpha V_b^a(k) \right\}, \quad (4)$$

where  $B_\mu$ ,  $B$ ,  $P$  and  $V$  describe the familiar  $SU(3)$ -multiplets and satisfy the appropriate free-field equations. In the rest-frame, the mass term (2) remains invariant under the group  $U(6) \otimes U(6)$  with the generators  $(1 \pm \gamma_4) \otimes \sigma_i \otimes \lambda_a$  [3].

For amplitudes with two or more independent momenta, the insertion of momentum dependent spurions  $S$  generally gives rise to new terms. However, in the case of Green's functions with *two independent momenta*, most substitutions are reducible and the  $SU(6)$  super-multiplet structure remains intact. We may choose a Lorentz-frame such that all spurions can be expressed in the form  $\gamma \cdot p = \gamma_4 p_4 + \gamma_3 p_3$ , and hence we have invariance under a  $U(6)$ -group with the generators [3]  $(1, \gamma_4 \sigma_1, \gamma_4 \sigma_2, \sigma_3) \otimes \lambda_a$ , which commute with  $\gamma_3$  and  $\gamma_4$ .

For amplitudes with *three* independent momenta, we can bring all spurions into the form  $\gamma \cdot p = \gamma_4 p_4 + \gamma_3 p_3 + \gamma_2 p_2$ , and there remains



invariance under the group  $U(3) \otimes U(3)$  with the generators  $(1 \pm \gamma_4 \sigma \cdot \hat{n}) \otimes \lambda_a$ , where  $\hat{n}$  is a unit vector in the 1-direction which is normal to the plane of scattering. Finally, with *four* or more momenta, the symmetry is broken down to  $U(3)$ .

It is important to note that in our scheme we have considered only the external momenta in a given channel of an amplitude. This is quite sufficient for vertex-functions, but in a scattering amplitude also crossed channels are relevant. Together with unitarity, these crossed channels can give rise to further symmetry breaking. We expect, therefore, that the spurion scheme works best for vertex-functions. In the case of reaction-amplitudes, it will presumably be necessary to supplement the symmetry with dispersion-theoretical considerations [9, 10].

## 2. MASS FORMULAE

So far, we have considered only momentum dependent spurions, but in second and higher orders we can also have more general insertions. For instance, in second order there are terms of the form

$$S_2 = \{S_1 \otimes 1 + S_P \gamma_5 \otimes \gamma_5 + S_A i \gamma_\mu \gamma_5 \otimes i \gamma_\mu \gamma_5 + \\ + S_V \gamma_\mu \otimes \gamma_\mu + S_T \sigma_{\mu\nu} \otimes \sigma_{\mu\nu}\} \otimes 1. \quad (5)$$

Although these spurions preserve Lorentz invariance and  $SU(3)$ -symmetry, they generally break the  $U(6)$  multiplets. Inserting  $S_2$  into the baryon mass term (2), we can take  $p = 0$  because of Lorentz invariance, and we find that the axial vector and the tensor spurion in Eq. (5) give a splitting between the octet of spin  $\frac{1}{2}$ -particles and the decuplet of spin  $\frac{3}{2}$ -particles which make up the **56**-supermultiplet described by the **364**-“tensor”  $\Psi$  in Eq. (3).

For the **36**-supermultiplet of mesons described by the **144**-“tensor”  $\Phi$ , we also obtain a mass-splitting between the pseudoscalar and the vector-meson nonets which is caused also by axial vector and tensor spurions. In addition, we see from Eq. (4) that the tensor-as well as the vector-spurion in Eq. (5) splits the masses of singlet and octet vector mesons, whereas the pseudoscalar and axial-vector terms have the same effect for the pseudoscalar mesons. The relevant terms in the mass relations are of the form  $\text{Tr}(\gamma_5 \bar{\Phi}(k)) \text{Tr}(\gamma_5 \Phi(k))$ , etc.

In order to obtain favorable mass-formulae for the mesons, we certainly want a singlet-octet splitting for ps-mesons which is in-

dependent of the SU(3)-symmetry breaking [2], but there should be no such splitting for vector-mesons if we want to obtain the  $\omega\phi$ -mixing corresponding to

$$\omega^0 = \sqrt{\frac{2}{3}}\omega_1 + \sqrt{\frac{1}{3}}\omega_8, \quad \phi^0 = \sqrt{\frac{1}{3}}\omega_1 - \sqrt{\frac{2}{3}}\omega_8. \quad (6)$$

Summing up, we find that the spurion

$$S_2(\mathbf{1}) = \{S_1 \otimes 1 + S_P \gamma_5 \otimes \gamma_5 + S_A i\gamma_\mu \gamma_5 \otimes i\gamma_\mu \gamma_5\} \otimes 1 \quad (7)$$

is just what we need for baryons as well as for mesons.

If we now proceed to the SU(3)-symmetry breaking terms, we find that the baryons and the mesons must be handled quite differently. For baryons we have to include a spurion  $S_2(\mathbf{8})$  which is given by Eq. (7) with 1 replaced by  $\lambda_8$ . Making use of the spurions  $S_2(\mathbf{8})$  and  $S_2(\mathbf{1})$ , we obtain the mass formula [1, 13]

$$M = M_0 + M_1 J(J+1) + M_2 Y + M_3 [I(I+1) - \frac{1}{4}Y^2], \quad (8)$$

which is satisfied very accurately by the physical masses. The axial vector term in  $S_2(\mathbf{8})$  effectively corresponds to  $(\sigma \otimes \sigma) \otimes 1$  if inserted into the baryon mass term (2) with  $\mathbf{p} = 0$ , and its presence is necessary in order to have  $M_3 \neq 0$  in Eq. (8).

The spurion  $S_2(\mathbf{8})$  cannot be used for SU(3)-breaking in the meson mass-term [14] if we want to have the relations [12, 15]

$$m_\rho^2 = m_\omega^2, \quad m_\phi^2 - m_{K^*}^2 = m_{K^*}^2 - m_\rho^2 = m_K^2 - m_\pi^2. \quad (9)$$

These can only be obtained by restricting the octet component of the spurion to

$$(1 \otimes 1) \otimes \lambda_8. \quad (10)$$

In addition, we have then also the familiar formula [15]

$$m_K^2 = \frac{3}{4}m_\eta^2 + \frac{1}{4}m_\pi^2 \quad (11)$$

for the ps-mesons.

### 3. WEAK AND ELECTROMAGNETIC INTERACTIONS

We see that the spurions of the form (5) give a reasonable description of the symmetry breaking. In the following, we restrict ourselves to the momentum dependent spurions  $S$  which leave the U(6)-supermultiplets undisturbed if inserted into the mass terms.

Let us first discuss the form factors of electromagnetic and weak interactions. We have described elsewhere a possible construction of universal weak- and electromagnetic interactions on the basis of algebras generated by the components of lepton and hadron currents [16, 17]. This method is based upon the assumption that the fundamental structure of these interactions is essentially determined by the physics in very small dimensions, which is assumed to exhibit  $\gamma_5$ -symmetry. We can, if we want, formulate the basic bare couplings explicitly in terms of quark fields  $\psi^A = \psi^{z,a}$ , where  $a$  is the SU(3)-index. We find, using the semileptonic interaction as an example,

$$\mathcal{L}_{\text{weak}} = -\frac{G}{\sqrt{2}} \bar{\psi}_A [i\gamma_\alpha(1+\gamma_5)A I_\alpha^\dagger]_{A'}^A \psi^{A'} + \text{h.c.}, \quad (12)$$

with

$$I_\alpha = i\bar{v}_e \gamma_\alpha(1+\gamma_5)e + i\bar{v}_\mu \gamma_\alpha(1+\gamma_5)\mu, \quad (13)$$

and

$$A = \frac{1}{2}[(\lambda_1 + i\lambda_2) \cos \theta + (\lambda_4 + i\lambda_5) \sin \theta], \quad (14)$$

or

$$A = \frac{1}{2}[(\lambda_1 + i\lambda_2) + (\lambda_4 + i\lambda_5)]. \quad (15)$$

The choice between Eqs. (14) and (15) depends upon our model: the hadron analog of the  $U_L(4)$  algebra generated by the components of the lepton current may be obtained from the total hadron current (Eq. (14)) or separately from the strangeness-changing and non-changing parts of this current (Eq. (15)). The corresponding bare electromagnetic coupling can be written in the form

$$\mathcal{L}_{\text{elm}} = -e \bar{\psi}_A \left[ i\gamma \cdot A \frac{1}{2} \left( \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right) \right]_{A'}^A \psi^{A'}. \quad (16)$$

In the real world, the basic interactions like (12) and (16) are modified by the strong couplings, for which we want to use our spurion scheme of broken  $U_{\mathcal{Q}}(12)$ -symmetry as a leading approximation. In Eqs. (12) and (16) the leptonic and the electromagnetic insertions transform like components of the representation **144** of  $U_{\mathcal{Q}}(12)$ . Correspondingly, we write the bare vertices for baryons and mesons like

$$\bar{\Psi}_{ABC}(p') \Psi^{ABC'}(p) [i\gamma_\alpha(1+\gamma_5)A I_\alpha^\dagger]_{A'}^C \quad (17)$$

etc., and as a first step we insert momentum dependent spurions in

arbitrary order. Let us consider only the baryon octet contained in the representation **364**. All possible substitutions of  $S$ -spurions can be reduced to those where  $i\gamma_\alpha(1+\gamma_5)$  is replaced by

$$[i\gamma \cdot q, i\gamma_\alpha(1+\gamma_5)] = 2i\sigma_{\alpha\beta}q_\beta - 2q_\alpha\gamma_5, \quad (18)$$

and by

$$\{i\gamma \cdot q, i\gamma_\alpha(1+\gamma_5)\} = -2q_\alpha - 2i\gamma_5\sigma_{\alpha\beta}q_\beta. \quad (19)$$

The commutator gives rise to *first class* terms, whereas the anti-commutator just gives the *second class* currents. Restricting ourselves to the substitution (18), we obtain the vertex structure

$$\frac{G}{\sqrt{2}} \bar{\Psi}_{ABC}(p') \Psi^{ABC'}(p) \{ [i\gamma_\alpha(1+\gamma_5)F(-q^2) + (i\sigma_{\alpha\beta}q_\beta - q_\alpha\gamma_5)G(-q^2)] \Lambda \}_{C'}^C, \quad (20)$$

where  $q = p - p'$ , and  $\Lambda$  is given by Eq. (14) or (15). Correspondingly, the electromagnetic vertex becomes

$$e \bar{\Psi}_{ABC}(p') \Psi^{ABC'}(p) \left[ (i\gamma_\alpha F_1(-q^2) + i\sigma_{\alpha\beta}q_\beta F_2(-q^2)) \times \right. \\ \left. \times \frac{1}{2} \left( \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right) \right]_{C'}^C. \quad (21)$$

Here the contribution from the anticommutator (19) vanishes because of gauge invariance or time-reversal invariance. For the electromagnetic form factors of the nucleons we obtain then the expressions

$$G_E^p(-q^2) = \left( 1 + \frac{q^2}{4m^2} \right) \left( F_1(-q^2) - \frac{q^2}{2m} F_2(-q^2) \right), \\ G_E^n(-q^2) \equiv 0, \\ G_M^p(-q^2) = -\frac{3}{2} G_M^n(-q^2) = \left( 1 + \frac{q^2}{4m^2} \right) \left( \frac{1}{2m} F_1(-q^2) + F_2(-q^2) \right). \quad (22)$$

The zeros of the form factors at  $q^2 = -4m^2$  cannot be compensated by poles of  $F_1$  and  $F_2$ , because  $G_M$  and  $G_E$  have different  $d/f$ -ratios, and at threshold we have the requirement that  $G_M = G_E/2m$ . For  $q^2 = 0$ , we have  $F_1(0) = 1$ , and hence the magnetic moment of the proton is given by  $\mu_p = 1 + 2mF_2(0)$ .

So far, the effect of meson couplings has not been considered ex-



plicity. But later, we show that the spurion theory, in combination with a meson pole model for the Sachs form-factors, gives the additional results [18]

$$\mu_p = \frac{2m}{\mu}, \quad \frac{G_M^p(-q^2)}{G_E^p(-q^2)} = \frac{\mu_p}{2m}, \quad \frac{\langle r_p^2 \rangle}{6} \approx \left( \frac{\mu_p}{2m} \right)^2, \quad \text{etc.}, \quad (23)$$

which are in good agreement with experiments.

For semileptonic interactions of nucleons, we have the familiar SU(6)-result  $G_A/G = -5/3$  at  $q^2 = 0$  [19]. All weak form factors can be easily expressed in terms of the functions  $F(-q^2)$  and  $G(-q^2)$  in Eq. (20), but here we do not want to discuss these details. Also nonleptonic decay of hyperons can be successfully described within the framework of our spurion scheme [20].

#### 4. STRONG INTERACTIONS

As an example for the effect of momentum dependent spurions on strong vertices, we consider briefly the meson-baryon vertex-function, restricting ourselves to the octet as far as the baryons are concerned. Using the meson-tensor given in Eq. (4), we obtain three irreducible terms which may be written in the form

$$i\bar{\Psi}_{ABC}(p')\Psi^{ABC'}(p) \left[ g_0 \Phi_C^C(q) + g_1 \left( 1 + \frac{i\gamma \cdot q}{\mu} \right)_D^C \Phi_C^D(q) + g_s \left( \frac{i\gamma \cdot P}{2m} \Phi(q) \right)_D^D \delta_C^{C'} \right]. \quad (24)$$

In this equation we have *not* restricted  $q^2$  to the meson mass shell, in view of its use in meson-pole dominance models. For instance, we can construct such a model for electromagnetic form factors by replacing  $g_0 V(q)$  by  $\frac{1}{2}(\lambda_3 + 1/\sqrt{3}\lambda_8)f(-q^2)$  with  $f(-q^2) \propto \mu^2/(q^2 + \mu^2)$ . Since the second term in Eq. (24) is proportional to  $1 + q^2/\mu^2$  and hence does not have the pole at  $-q^2 = \mu^2$ , we may want to neglect it. The third term vanishes, and in comparison with Eq. (21), we have then

$$F_1(-q^2) \approx \mu F_2(-q^2) \propto f(-q^2), \quad (25)$$

which gives rise to the expression  $\mu_p = 1 + 2m/\mu$  for the magnetic moment of the proton.

On the other hand, we may bring Eq. (24) into the familiar form

involving Sachs form-factors. If we then require pole dominance for these form-factors, we find the result (23) [21].

We can also consider Eq. (24) as a vertex on the mass shell. Then the second term vanishes and the third one gives rise to a coupling like

$$g_s \frac{e \cdot P}{2m} \text{Tr} (\bar{B}B)(\sqrt{2}\omega^0 + \varphi^0), \quad (26)$$

which involves only the singlet vector meson. Hence, except for this singlet term, our spurion scheme gives a unique coupling between the baryon octet and the mesons.

For reaction-amplitudes involving four particles, the inclusion of momentum dependent spurions generally gives rise to several new terms. This is because we are left with the lower symmetry  $U(3) \otimes U(3)$ . Of special interest are reactions of the type ps-meson + baryon  $\rightarrow$  ps-meson + baryon. Here we obtain, in the limit of formal  $U_{\mathcal{A}}(12)$ -invariance, the prediction of zero polarization for reactions like [8]

$$\begin{aligned} K^- + p &\rightarrow \bar{K}^0 + n, & \pi^- + p &\rightarrow K^+ + \Sigma^-, \\ K^- + p &\rightarrow \pi^+ + \Sigma^-, & K^+ + n &\rightarrow K^0 + p, \end{aligned} \quad (27)$$

and also for

$$K^- + p \rightarrow K^+ + \Xi^-. \quad (28)$$

The same is true, of course, for reactions which are related to those in Eqs. (27) and (28) through isospin invariance or  $SU(3)$ -invariance; for example, the processes  $\bar{K}^0 + n \rightarrow K^0 + \Xi^0$  and  $K^- + p \rightarrow K^0 + \Xi^0$  are related to reaction (28) in this way.

Let us write the amplitudes for all these processes in the familiar form

$$\bar{u}(p') \left\{ A(s, t) - i\gamma \cdot \frac{k+k'}{2} B(s, t) \right\} u(p). \quad (29)$$

Without momentum-spurions, we find  $\text{Im}(AB^*) = 0$  and  $B(s, t) = 0$  for reactions (27) and (28), respectively. We have four amplitudes which are given by

$$\begin{aligned} \bar{\Psi}_{ABC} \Psi^{ABC} (\bar{\Phi}\Phi)_D^D f_1(s, t) + \\ + \bar{\Psi}_{ABC} \Psi^{ABC'} \{ (\bar{\Phi}\Phi)_{C'}^C f_2(s, t) + (\Phi\bar{\Phi})_C^C \bar{f}_2(s, t) \} + \\ + \bar{\Psi}_{ABC} \Psi^{AB'C'} \bar{\Phi}_B^B \Phi_{C'}^C f_3(s, t). \end{aligned} \quad (30)$$

Insertion of the spurion  $S$  in first order gives rise to eight additional terms; four terms result from the contraction of the spurion indices with those of the baryon tensors, and the remaining four involve also contractions with indices of the meson tensors. In all cases, we find that we can restrict ourselves to the term

$$i\gamma \cdot (k+k') \otimes 1 \quad (31)$$

in the expansion (1) of the spurion  $S$ , other invariants being reducible. With the inclusion of the spurion terms, it is easy to see that polarization effects become possible for the reactions (27). However, the only invariant which gives rise to a finite  $B$ -coefficient in Eq. (29) for the process (28) is given by

$$\bar{\Psi}_{ABC}(p')[i\gamma \cdot (k+k')]_{A'}^A \Psi^{A'B'C'}(p) \bar{\Phi}_B^B(k') \Phi_C^C(k). \quad (32)$$

Experimentally we know that the polarization of the  $\Xi^-$  in the reaction (28) is large over a wide range of angles [21], and this seems to be a rather strong indication that spurion-terms are needed [9].

Of special importance are the implications of the spurion-scheme for forward scattering amplitudes. It is easy to see in general that in this case the substitution of  $S$ -spurions does not give rise to new amplitudes [9]. Hence the relations [22]

$$\begin{aligned} F(\pi^+ p) - F(\pi^- p) &= F(K^0 p) - F(\bar{K}^0 p) \\ &= \frac{1}{2}\{F(K^+ p) - F(K^- p)\} \end{aligned} \quad (33)$$

between the elastic forward scattering amplitudes are preserved in our theory.

Of interest are also the predictions of the spurion scheme for the amplitudes describing the annihilation of nucleon-antinucleon pairs at rest. Due to the orthogonality of the spinor wave functions of a particle and an antiparticle with equal momenta, we have the limit

$$\lim_{p \rightarrow 0} \bar{\Psi}_{ABC}(-p) \Psi^{AB'C'}(p) = 0. \quad (34)$$

Assuming that these zeros are not cancelled by artificial poles of the coefficients, Eq. (34) implies that, without momentum spurions, there are no annihilations into two mesons [23]. With a spurion  $S$  in first order, we could have the amplitude

$$\lim_{p \rightarrow 0} g \cdot \bar{\Psi}_{ABC}(-p) (i\gamma \cdot k)_{A'}^A \Psi^{A'B'C'}(p) \bar{\Phi}_B^B(k') \bar{\Phi}_C^C(k), \quad (35)$$

with  $\mathbf{k} = -\mathbf{k}'$  being the c.m.-momentum of a meson in the final state. However, it is easy to see that the expression (35) violates charge-conjugation invariance. For instance, it would give rise to processes like  $p + \bar{p} \rightarrow \pi^0 + \pi^0$  or  $p + \bar{p} \rightarrow \eta + \eta$ , which are forbidden by  $C$ -invariance for annihilations at rest. Hence we must have  $g = 0$ .

If all higher orders are included, the only non vanishing amplitude is

$$\lim_{p \rightarrow 0} g' \bar{\Psi}_{ABC}(-p)(i\gamma \cdot k)_A^A (i\gamma \cdot k)_{B'}^B \Psi_{(p)}^{A'B'C'} \{ \bar{\Phi}(k') \bar{\Phi}(k) + \bar{\Phi}(k) \bar{\Phi}(k') \}^C_{C'}. \quad (36)$$

However, this amplitude does not appear in a pole dominance model, and hence, within this framework, two meson annihilation is forbidden by our spurion scheme. It is possible that this result is responsible for the empirical fact that two-meson annihilation is suppressed compared to three- and four-meson annihilations.

There are many consequences of the spurion scheme which remain to be worked out. However, from the discussions given above, we see already that the scheme is very successful for vertex-functions. As we have pointed out, the application of our scheme to reaction amplitudes is quite a different problem. But the spurion scheme is completely Lorentz-covariant, and it is sufficiently flexible to allow for the inclusion of additional dynamical considerations.

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# CAUSALITY AND DISPERSION RELATIONS

*(A Dialogue on Classical Physics)*

R. HAGEDORN

*CERN, Geneva*

*(Received May 4, 1965)*

Persons: Inventor – *I*

Physicist – *P*

*I*: Hallo, old boy! Stop dreaming and come down to earth – remember me?

*P*: – . . . ?

*I*: Of course you don't. We studied physics together long ago, but after three years of learning I found the stuff too abstract and gave it up. Now I . . .

*P*: Indeed, yes, I remember. Glad to see you again; you seem to be very well. What are you doing?

*I*: Inventing all sorts of things in which a little knowledge of physics helps me very much. It's a hobby which earns me a good living. Just now I have made a really fantastic invention. Imagine, I am going to make sunglasses with which one can see in the dark.

*P*: How's that?

*I*: Very simple; so simple that I wonder why I am the first to think of it. Probably because the other inventors do not know enough physics and because the physicists are not practical minded enough. That abstract thinking of yours does no good.

Anyway, the idea is this: suppose you are in a dark room in which there is an electronic flash light – something of the sort used in your bubble chambers. The flash light will be operated at any time after you enter, but presently the room is absolutely dark. However, if you Fourier-analyse the flash, it will contain all frequencies

and each of them will be present as an everlasting sinuswave. Their amplitudes and phases are arranged so that by superposing them they cancel each other out except for that small fraction of a second where the flash interrupts the darkness. You agree?

*P:* Of course, go on.

*I:* Now, my invention is to use a pair of spectacles with a coloured glass which lets through only one frequency – or a very small frequency interval – and absorbs the rest. This one frequency belongs to an everlasting sinus-wave. The rest of the spectrum which is absorbed, can no longer help to cancel this wave and thus your eyes receive light and you can see. Isn't that great?

*P:* Sure. Did you try it?

*I:* I did, but so far, for some reason or other, it did not work. Maybe I have not yet found the right glass; or maybe the frequency interval which goes through is still too large. I shall try a very selective filter now. Perhaps, then, the energy contained in that small frequency band is also very small and I must add some amplifying device – but these are technical details.

*P:* Maybe you forgot another less technical detail and I have some idea about it. But now I am in a hurry. Let us meet tomorrow night in that Coffee bar over there at eight-o'clock, right?

*I:* Fine. And I promise you, I shall not be a fraction of a second late. So long!

*P:* So long.

*Next evening at eight fifteen*

*I:* Hallo, sorry to be so late.

*P:* You promised not to be late.

*I:* I am really sorry. I started from home an hour ago although the way from there to here is hardly more than twenty minutes. Unfortunately I just missed the bus. I would still have arrived much

too early with the next one; but as soon as I boarded it, the engine broke down – and you know, that sort of thing happens only once in ten years. So I decided to walk and I would still have arrived in time had a man not tried to steal some jewellery by throwing a stone into a window just as I passed. He was obviously mad – imagine! to rob a jewellery store in the early evening in the midst of all those people! Of course they got him – and how mad he was! Not a single sensible word did he say though he was talking all the time. As I had been unlucky enough to see how things happened, they took me to the police and I had to tell them every detail a dozen times. Now, finally, here I am.

*P:* But you promised not to be late.

*I:* I tried my best, as I told you. But now how about the detail which you think I overlooked?

*P:* It's a pity you are so late.

*I:* You are boring me with your reproaches. Could I look into the future? Could *you* have foreseen the extremely improbable chain of events which made me so late in arriving? Nobody could, I tell you!

*P:* And that is just the little detail which makes your invention fail to work.

*I:* How's that? If you want to fool me I had better go now.

*P:* Wait. You started from Fourier-analysing a flash of light. Now, Fourier-analysis applies to everything varying in time. Can you think of another example?

*I:* That is a good idea. Maybe my invention can be extended to other cases, let's see. Yes, indeed, to sound.

*P:* I thought of that. Hold a tuning fork next to a gun. What happens?

*I:* According to my argument it should vibrate, because it selects one single Fourier component out of the bang. The principle is the same as with my coloured glasses.

*P:* Indeed. And did you ever observe something like this?



*I:* Of course I did. I remember well my experiments with oxyhydrogen gas; a tuning fork happened to stay in my laboratory and it gave a sound still 20 seconds after an experimental explosion.

*P:* And sure, you know also substances which give light after having been exposed to a flash?

*I:* Why are you asking me such questions? Everybody knows them from his wrist-watch. And your examples only prove that my invention *must* work. I begin to doubt, however, whether it will work very well practically. The point is that phosphorescent materials, tuning forks, and similar mechanisms, emit with decreasing intensity and not for a long time. Do you mean this is your detail?

*P:* Not quite. Tell me, did your tuning fork sound also *before* your experimental explosion?

*I:* No, of course not! Because . . . – oh, did I say “of course”? Of course I do *not* mean “of course” – you only caught me unawares. What I really mean is . . . –

*P:* ?

*I:* Wait . . . – – – it is true, I never observed such a thing. On the one hand this seems to be natural – that is why the “of course” escaped me; but it contradicts my argument of Fourier-analysis and filtering out one frequency. And therefore it seems by no means “of course”. There is a contradiction which I really do not understand – but you not only caught me unawares, you really seem to have made a point here.

*P:* Think of your tuning fork. What if it really had started sounding before the explosion?

*I:* I see what you are driving at. With its damping time of about 20 seconds it should have started sounding about 20 seconds before the explosion – but how could it have known exactly *when* the bang would occur?

*P:* If even you could not foresee that tonight you would be too late! Do you see now the missing detail?

- I*: I start seeing it. Do you say my invention – at least my argument – violates causality? Do you say, that if it would work, then from observing that the tuning fork starts to oscillate I could conclude that the explosion will take place with *certainty* within the next minute – in spite of the possibility that the ignition may fail?
- P*: Your answer is correct and you found it yourself.
- I*: I must confess that this general argument does not enlighten me very much – although I must accept it. It has almost the same convincing force on me as the postulate of the conservation of energy or the second law of thermodynamics has on a man who just invented a perpetual mobile. Have you ever heard of such a man who says “Thank you, Sir, for reminding me of that general law. I now see clearly that my invention cannot work.” No perpetual mobilist will give in before you point out the specific fault of his machine. I am not so stupid as to fight against causality as long as there are no serious reasons to doubt it, but the argument does not satisfy me. I must see where my consideration fails. Did you not agree that each Fourier component is an everlasting oscillation and that there are mechanisms (filters) which are able to select one (or a few) frequencies? What is wrong, then?
- P*: We shall find out. Let us try to describe things as generally as possible (to prevent you, my friend, from coming back tomorrow with a new method to deceive causality). You have a system, called black box, which stands for everything of the kind we are discussing (filter glass, tuning fork etc.). There is some force, called input, acting on the box and there is some response from the box; we call that output. Both input and output are functions of time. Now please will you take over and describe the properties of that box – I mean properties which all such boxes share if they can be called causal boxes.
- I*: O.K. First of all, I think, this black box should relate input and output linearly. I know of non-linear systems which can excite themselves.
- P*: Well, this might perhaps be a too strong restriction, because we

also know non-linear systems which do not excite themselves and which are causal. But let us assume linearity for simplicity. What else?

- I:* Your black box should remain always identical to itself; if it had internal properties which can change in the course of time, it would also be able to emit an output without having received an input. This output would signalize that something inside the box is happening.
- P:* Is that all? So far you only excluded that the black box acts on its own account. What about the causal relation between a given input and the corresponding output? Think of the tuning fork!
- I:* If I imagine a bang or a flash – I mean any input of the form  $f(t) = \delta(t - t_0)$  – then, accepting your causality argument, I would require that no output  $g(t)$  can start earlier than  $t_0$ , but may last some time after  $t_0$ .
- P:* Very good. Now let us formulate that mathematically. That the output is a linear functional of the input may be expressed by writing

$$g(t) = \int_{-\infty}^{+\infty} L(t, t') f(t') dt'. \quad (1)$$

you agree?

- I:* Quite. And I see more. If  $L(t, t')$  expresses what the black box does, and if the black box must have no properties which change in time, then  $L(t, t')$  should depend only on the difference  $t - t'$ .
- P:* And your last requirement?
- I:* Well, with  $f(t) = \delta(t - t_0)$  we find

$$g(t) = \int_{-\infty}^{+\infty} L(t - t') \delta(t' - t_0) dt' = L(t - t_0)$$

and  $g(t)$  must be zero for  $t < t_0$ , hence

$$L(t - t_0) \equiv 0 \quad \text{for } t < t_0. \quad (2)$$

I do not see, however, the relation to my problem. I started from

considering what the black box does on each single Fourier component.

*P*: So why do you not Fourier-transform the equation

$$g(t) = \int_{-\infty}^{+\infty} L(t-t')f(t')dt' \quad (3)$$

and see what comes out?

*I*: Good idea. I write the Fourier-transform

$$\tilde{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{i\omega t} dt$$

and find

$$\tilde{g}(\omega) = \tilde{L}(\omega) \cdot \tilde{f}(\omega). \quad (\tilde{3})$$

This is very nice. A monochromatic input is simply multiplied by a number. If the box acts as a filter, then  $\tilde{L}(\omega)$  is zero except in the frequency range which is filtered out. I still do not see what is wrong with my invention – on the contrary, this formula seems to support my argument.

*P*: You have not yet fully exploited causality – I mean the fact that  $L(t-t') \equiv 0$  for  $t-t' < 0$ . You will be surprized what this condition teaches you on  $\tilde{L}(\omega)$ .

*I*: Let's see. I write

$$\tilde{L}(\omega) = \int_{-\infty}^{+\infty} L(\tau)e^{i\omega\tau} d\tau; \quad L(\tau) \equiv 0 \text{ for } \tau < 0.$$

Yes, I see:  $\tilde{L}(\omega)$  is an analytic function of  $\omega$  and it is holomorphic in the upper half plane, as only positive  $\tau$  contribute to the integral. You mean, this seriously limits the possibilities of  $\tilde{L}(\omega)$ ?

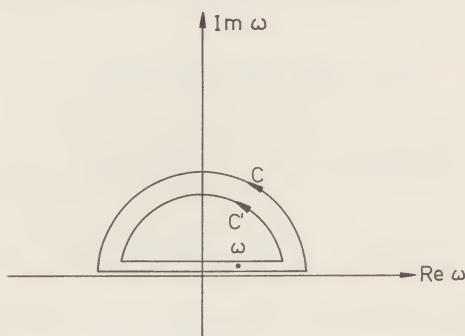
*P*: Yes. And now let me use a little trick to obtain a more detailed description of this limitation; I should better say: another description, because it is equivalent to  $\tilde{L}(\omega)$  being holomorphic in the upper half plane. As I know you, you will not insist on mathematical subtlety and that makes it easy to explain the main points rather shortly. As  $\tilde{L}(\omega)$  is holomorphic in the upper half plane,



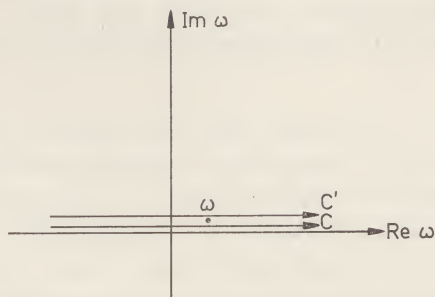
its value for  $\text{Im } \omega > 0$  is given by Cauchy's formula as

$$\tilde{L}(\omega) = \frac{1}{2\pi i} \int_C \frac{\tilde{L}(\omega')}{\omega' - \omega} d\omega' + \frac{1}{2\pi i} \int_{C'} \frac{\tilde{L}(\omega')}{\omega' - \omega} d\omega', \quad (4)$$

where the following figure shows the paths of integration. The



second integral contributes nothing. My trick is just to add it, nevertheless. Now let the radii of the half circles go to infinity and assume that  $\tilde{L}(\omega)$  decreases fast enough so that these half circles give no contribution in that limit (if this should not be the case, we consider  $\tilde{L}(\omega)$  divided by a suitable polynomial instead of  $\tilde{L}(\omega)$  itself). Then we remain with the following paths:



$C$  and  $C'$  extend now from  $-\infty$  to  $+\infty$ . Finally we let  $C$  and  $C'$  approach the real axis, always keeping  $\omega$  between them and obtain...

*I:* ... twice the principal value integral, isn't it?

*P*: Yes, you remember well, we obtain [P indicates principal value]

$$\tilde{L}(\omega) = \frac{1}{2\pi i} \cdot 2 \cdot P \int_{-\infty}^{+\infty} \frac{\tilde{L}(\omega')}{\omega' - \omega} d\omega' \quad [\omega \text{ real}].$$

Of course, this is no proof; our heuristic argument leads, however, to the correct formula. We now write the real and the imaginary parts separately:

$$\tilde{L}(\omega) = \text{Re } \tilde{L}(\omega) + i \text{Im } \tilde{L}(\omega)$$

then

$$\begin{aligned} \text{Re } \tilde{L}(\omega) &= \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Im } \tilde{L}(\omega')}{\omega' - \omega} d\omega' \\ \text{Im } \tilde{L}(\omega) &= -\frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{\text{Re } \tilde{L}(\omega')}{\omega' - \omega} d\omega'. \end{aligned} \quad (5)$$

Now this pair of formulae – called dispersion relations – is fully equivalent to  $L(\tau) \equiv 0$  for  $\tau < 0$  and thus to causality. Of course, there are some fine mathematical points which we neglected. They are contained in the exact formulation and proof of Titchmarsh's theorem, which states (loosely speaking) that for a function  $\tilde{L}(\omega)$  the three properties:

( $\alpha$ ) obeying dispersion relations

( $\beta$ ) having a Fourier-transform  $L(\tau)$  vanishing for  $\tau < 0$

( $\gamma$ ) being holomorphic in the upper half plane

are in fact only one single property expressed three times in different words.

*I*: I start to see how it works. But what about the examples: the optical filter and the tuning fork? I remember well that in classical electrodynamics the optical properties of transparent matter can be made plausible by relating the refractive index to the electric polarizability and calculating the latter from a model in which electrons are elastically bound and oscillate according to the external electric field of the light source. And the tuning fork, after all, is also something like a linear oscillator – could we not try to discuss just a particular black box, namely a damped linear oscillator and try all our considerations on it? I, for my part, shall not be able to sleep before I have seen this practical example solved.

*P*: Very well. Let us write

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f(t) \quad (6)$$

and consider  $f(t)$  as the input and  $x(t)$  as the output. We take the Fourier-transform of this equation;

$$x(t) = \frac{1}{2\pi} \int \tilde{x}(\omega) e^{-i\omega t} d\omega$$

$$f(t) = \frac{1}{2\pi} \int \tilde{f}(\omega) e^{-i\omega t} d\omega.$$

That gives

$$\tilde{x}(\omega) = \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \cdot \tilde{f}(\omega). \quad (7)$$

Comparing with equation (3) shows, that our black box in this case is described by

$$\tilde{L}(\omega) = \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} = - \frac{1}{(\omega - \omega_1)(\omega - \omega_2)} \quad (8)$$

where  $\omega_1$  and  $\omega_2$  are ...

*I*: Let me try to continue. This function should be holomorphic in the upper-half plane. Indeed, it has only two poles at  $\omega_1$  and  $\omega_2$ , namely

$$\omega_{1,2} = -\frac{1}{2}i\gamma \pm \omega'_0 \quad \text{with } \omega'_0 = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2} \quad (9)$$

and both lie in the lower-half plane. This should then imply that the function  $L(\tau)$  is causal, i.e. vanishes for  $\tau < 0$ . To show that we have to calculate the integral

$$L(\tau) = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega\tau}}{(\omega - \omega_1)(\omega - \omega_2)} d\omega.$$

Now for  $\tau < 0$  we may displace the path from  $-\infty$  to  $+\infty$  by shifting it parallel to  $+i\infty$ . Since the integrand has no singularities in the upper-half plane the integral vanishes.

On the other hand, for  $\tau > 0$  we must shift the path to  $-i\infty$  if we wish to have the integrand vanish. In that case the residues of

the poles contribute

$$L(\tau) = \frac{1}{2\pi} \cdot 2\pi i \left[ \frac{e^{-i\omega_1\tau}}{\omega_1 - \omega_2} + \frac{e^{-i\omega_2\tau}}{\omega_2 - \omega_1} \right] = \frac{1}{\omega'_0} e^{-\frac{1}{2}\gamma\tau} \sin \omega'_0\tau$$

thus

$$L(\tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ \frac{1}{\omega'_0} e^{-\frac{1}{2}\gamma\tau} \sin \omega'_0\tau & \text{for } \tau > 0. \end{cases}$$

If we now write

$$\begin{aligned} x(t) &= \int_{-\infty}^{+\infty} L(t-t')f(t')dt' \\ &= \frac{1}{\omega'_0} \int_{-\infty}^t e^{-\frac{1}{2}\gamma(t-t')} \sin [\omega'_0(t-t')]f(t')dt' \end{aligned}$$

then we see explicitly the causal behaviour. That  $\tilde{L}(\omega)$  fulfils the dispersion relations equations (5), I will believe without checking it.

*P:* To complete your consideration, you should discuss the bang or flash – I mean  $f(t') = \delta(t')$ !

*I:* Well, quite generally – as we have already seen –

$$x(t) = \int_{-\infty}^{+\infty} L(t-t')\delta(t')dt' = L(t)$$

where  $L(t) \equiv 0$  for  $t < 0$ . Say – isn't that just what one calls the Green's function of the differential equation? That solution where the inhomogeneous part of the equation is a  $\delta$ -function?

*P:* Quite so; but now discuss the explicit form!

*I:* Here it is; let us call it  $x_0(t)$ :

$$x_0(t) = \begin{cases} 0 & \text{for } t > 0 \\ \frac{1}{\omega'_0} e^{-\frac{1}{2}\gamma t} \sin \omega'_0 t & \text{for } t \geq 0. \end{cases} \quad (10)$$

The oscillation starts at  $t = 0$  with frequency  $\omega'_0$  and a damping constant  $\frac{1}{2}\gamma$  and these two numbers are just (up to the sign) the



imaginary and the real part of the poles of  $\tilde{L}(\omega)$ . The amplitude with which the oscillation starts, namely  $1/\omega'_0$ , equals just the sum of the moduli of the residues of  $\tilde{L}(\omega)$  at the poles [see (8) and (9)].

- P:* Let me summarize the results: A causal system is well able to select some part of a spectrum and to (almost completely) absorb the rest – but the real and the imaginary parts of  $\tilde{L}(\omega)$  are always arranged in such a way that no matter what part of the spectrum is absorbed, the rest gets just the right phase shifts so that no output can precede the input. The dispersion relations express this relation between the real and the imaginary part. The output may, however, be delayed with respect to the input. How much and with what amplitude and “lifetime” that depends on the location and residues of the singularities of  $\tilde{L}(\omega)$  in the lower-half plane. You will find similar situations in all cases where causality is involved – for instance in the quantum theory of scattering, where  $\tilde{L}(\omega)$  becomes more complicated. There it is called the scattering amplitude; it will have not only poles but also cuts. The cuts are related to the production of particles in a scattering process but the poles have a significance very similar to that found in our simple example: their real and imaginary parts are the frequencies (= energies) and inverse lifetimes of resonances.
- I:* Thank you very much – I now see that causality is not only a general argument against my poor invention; I even see how material physical systems manage to reconcile causality with the existence of frequency filters. I certainly shall not try to find a better filter glass! Good-bye!

*P:* Good-bye!

# DIE ROLLE DER PHÄNOMENOLOGISCHEN THEORIEN IM SYSTEM DER THEORETISCHEN PHYSIK

W. HEISENBERG

*München*

*(Eingegangen am 5. Mai 1965)*

Unter „phänomenologischer“ Theorie kann man die Formulierung von Gesetzmäßigkeiten im Bereich der beobachteten physikalischen Phänomene verstehen, bei denen nicht versucht wird, den zu beschreibenden Zusammenhang auf ein zugrunde liegendes allgemeines Naturgesetz zurückzuführen und dadurch verständlich zu machen. Solche phänomenologischen Theorien haben in der Entwicklung der Physik immer wieder eine bedeutende Rolle gespielt; für die technischen und sonstigen Anwendungen mögen sie oft wichtiger sein als das Verständnis der Zusammenhänge, und für eine rein pragmatische Einstellung kann die phänomenologische Theorie die Kenntnis der Naturgesetze sogar weitgehend überflüssig machen.

Phänomenologische Theorien entwickeln sich begreiflicherweise immer dort, wo die beobachteten Erscheinungen noch nicht auf allgemeine Naturgesetze zurückgeführt werden können. Der Grund für diese Unmöglichkeit kann entweder in dem hohen Komplikationsgrad der betreffenden Erscheinungen liegen, der eine solche Zurückführung wegen der mathematischen Schwierigkeiten noch nicht gestattet, oder in der Unkenntnis der betreffenden Naturgesetze selbst. Beispiele für den ersten Fall sind etwa in der Meteorologie die halb empirischen Gesetzmäßigkeiten, die für die Wettervorhersage benützt werden, in der Chemie die Valenzregeln oder die Zusammenhänge zwischen Atom- und Ionenradien, Bindungs- und Aktivierungsenergien usw., in der Strömungslehre die Beziehungen zwischen Geschwindigkeit, Strömungswiderstand, Wärme- und Impulsaustausch bei der turbulenten Bewegung usw. Beispiele für den zweiten Fall sind in der Optik um die Jahrhundertwende die Formeln der Drudeschen Dispersionstheorie oder die empirischen Regeln über die Optik bewegter Körper;

in der ersten Hälfte des 19. Jahrhunderts die Überlegungen Faradays zur Elektrizitätslehre und die phänomenologische Thermodynamik, in der antiken Astronomie die Ptolemäische Zyklen- und Epizyklen-theorie der Planetenbewegung.

Der wichtigste gemeinsame Zug dieser phänomenologischen Theorien besteht darin, daß sie zwar eine zutreffende Beschreibung der beobachteten Erscheinungen ermöglichen, daß sie insbesondere oft eine sehr genaue Vorausberechnung neuer Experimente oder späterer Beobachtungen erlauben, daß sie aber doch kein eigentliches Verständnis der Erscheinungen vermitteln. Es soll hier nicht versucht werden, den Begriff „eigentliches Verständnis“ näher zu definieren. Denn man erfährt oft erst durch die Entwicklung der Wissenschaft, was das Wort Verständnis bedeutet. Aber dieses „eigentliche Verständnis“ unterscheidet sich grundsätzlich und qualitativ von dem Inhalt der phänomenologischen Theorie, wie man am besten an den angeführten Beispielen erkennen kann: Die Bewegungen der Planeten hat man erst mit Kopernikus, Kepler und der Newtonschen Physik wirklich verstanden; die Gesetze der Chemie erst mit der Bohrschen Atomtheorie und der Quantenmechanik. Noch ein spezielleres Beispiel soll hier angeführt werden: Der recht komplizierte anomale Zeeman-Effekt der D-Linien des Natriumatoms war schon 1912 von W. Voigt mit dem Modell der gekoppelten Oszillatoren in allen Einzelheiten richtig beschrieben worden. Aber erst 15 Jahre später konnte der Sinn der Voigtschen Formeln aufgrund der Quantentheorie richtig verstanden werden.

Wenn der erste der beiden genannten Fälle zutrifft, d.h. wenn nur der Komplikationsgrad und die aus ihm resultierenden mathematischen Schwierigkeiten eine Zurückführung der Erscheinungen auf Naturgesetze verhindern, so ist die phänomenologische Theorie ein Notbehelf, der im Hinblick auf die praktischen Anwendungen sehr wichtig und nützlich sein kann. Interessanter ist aber der zweite Fall, in dem die zugrunde liegenden Naturgesetze noch gar nicht bekannt sind. Hier wird man hoffen, daß die phänomenologischen Theorien den Weg zur richtigen Formulierung der Naturgesetze weisen könnten, und man wird an dieser Stelle nach dem heuristischen Wert der phänomenologischen Theorie fragen.

Zunächst wird man feststellen, daß man zwei deutlich getrennte

Arten von phänomenologischen Theorien unterscheiden kann. Die einen, die im wesentlichen formale Zusammenhänge ausnützen, und die anderen, die qualitativ und oft noch unklar das formulieren, was man – mit einem bewußt unbestimmten Ausdruck – als das „physikalisch Wesentliche“ bezeichnet. Die erwähnte Voigtsche Theorie des anomalen Zeeman-Effekts hat rein formale Zusammenhänge ausgenützt, allerdings mit erstaunlichem Erfolg; aber sie hat die Phänomene nicht erklärt. Ein anderes Beispiel von sehr viel größerem Gewicht ist zwei Jahrtausende früher die Astronomie des Ptolemäus gewesen; sie hat die rein formale Möglichkeit ausgenützt, periodische Bewegungen durch Fourierreihen darzustellen. Die phänomenologische Thermodynamik des 19. Jahrhunderts dagegen hatte mit der Formulierung des Entropiebegriffs etwas „physikalisch Wesentliches“ gefunden, ebenso die Chemie mit der Aufstellung der Valenzregeln. Offenbar ist der heuristische Wert der Theorien der ersten Gruppe verhältnismäßig gering, da der formale Zusammenhang eben das Wesentliche oft nicht erkennen läßt. Dagegen sind die phänomenologischen Theorien der zweiten Gruppe in der Regel die Vorstufen zum endgültigen Verständnis. Aus der Darstellung der Planetenbewegung durch Zyklen und Epizyklen in der Astronomie des Ptolemäus konnte man über den inneren Zusammenhang dieser Bewegungen fast nichts lernen. Die Keplerschen Gesetze jedoch sind die unmittelbare Vorstufe zur Newtonschen Mechanik.

Man erkennt an dieser Stelle auch, daß der Physiker oder Astronom schon unbewußt phänomenologische Theorien recht verschieden bewerten wird, je nachdem er in seiner philosophischen Einstellung durch den Pragmatismus geformt oder von anderen Gedankengängen, etwa der Ideenlehre Platons, beeinflusst ist. Wer im Pragmatismus aufgewachsen ist, wird eine phänomenologische Theorie um so höher bewerten, je mehr Erfolge sie aufweisen kann, je genauere Voraussetzungen sie zu geben gestattet. Wer jedoch schon früh von der Überzeugungskraft des platonischen Denkens ergriffen worden ist, wird die phänomenologischen Theorien vor allem danach beurteilen, ob und wie weit sie zum Verständnis der eigentlichen Zusammenhänge führen können. Die Entwicklung der Naturwissenschaft wird also an dieser Stelle entscheidend von dem in dem betreffenden Zeitalter oder Kulturkreis herrschenden philosophischen Denken bestimmt. In der



Antike war die Vorstellung, daß die Sonne im Mittelpunkt des Planetensystems steht, schon mehrfach ausgesprochen worden. Wenn sich trotzdem in der Spätantike die Ptolemäische Lehre durchgesetzt hat, so kann dies wohl nur bedeuten, daß in der philosophischen Haltung der Menschen jener Zeit das pragmatische Denken gegenüber dem prinzipiellen Denken der früheren Jahrhunderte die Oberhand gewonnen hatte. Eine erfolgreiche, aber doch nur formale phänomenologische Theorie hat in der Folge dann für anderthalb Jahrtausende den Weg zu einem echten Verständnis der Planetenbewegung versperrt.

Wendet man solche Überlegungen auf die heutige Physik an, insbesondere auf die jetzt im Mittelpunkt des Interesses stehende Theorie der Elementarteilchen, so lernt man aus den genannten Beispielen, wie wichtig es ist, den entstehenden oder schon entstandenen phänomenologischen Theorien anzusehen, ob sie den Weg zu einem Verständnis der eigentlichen Zusammenhänge weisen, oder ob sie mehr formaler Natur sind. Dafür gibt es allerdings keine allgemein brauchbaren Kriterien, und bis zur endgültigen Klärung der Zusammenhänge werden verschiedene Physiker die einzelnen phänomenologischen Ansätze verschieden beurteilen. Es sollte an dieser Stelle nur noch einmal auf die Bedeutung jenes vom Pragmatismus etwas vernachlässigten Bereiches der theoretischen Physik hingewiesen werden, der mit dem Ausdruck „physikalisches Verständnis“ nur sehr unvollkommen charakterisiert werden kann. Er ist hier hervorgehoben worden, weil er auch von Weisskopf oft zum Ausgangspunkt seiner physikalischen Überlegungen gemacht worden ist.

# THE CONCEPT OF MAXIMAL $CP$ VIOLATION

LINCOLN WOLFENSTEIN

*Carnegie Institute of Technology, Pittsburgh, Pennsylvania*

*(Received May 5, 1965)*

The observation [1] that the long-lived  $K^0$  meson decays into two pions provides an apparent violation of  $CP$  invariance in weak interactions. While the magnitude of the observed effect seems to indicate that the violation is small, it is not really possible to relate the observation in any quantitative way to the  $CP$  violation in the interaction Hamiltonian. In this respect the situation resembles the first case of parity violation, which amusingly also involved the pionic decays of the  $K$  meson. There was no way (and indeed to this day is no way) to relate the observed ratio of  $3\pi$  to  $2\pi$  decays of the  $K^+$  meson to the amount of parity violation in the weak interaction Hamiltonian. The difficulty arises from the fact that in a transition to a state of strongly-interacting particles there is no direct simple relation between the observable decay amplitudes and the interaction Hamiltonian. For example, the observation of a maximum value of a parity-violating effect, as in the decay asymmetry of  $\Sigma^+ \rightarrow \pi^0 + p$ , does not indicate a maximum violation of parity in the Hamiltonian any more than the small values of the asymmetry in the decays  $\Sigma^+ \rightarrow \pi^+ + n$  and  $\Sigma^- \rightarrow \pi^- + n$  indicate that parity violation in the Hamiltonian is small. When processes involving leptons were considered, however, it was possible to develop a clear-cut concept of maximal parity violation, which has proven to be a good description of the observations on muon decay, pion decay, and nuclear  $\beta$ -decay. We wish in this note to develop the concept of maximal  $CP$  violation defined as similarly as possible to the accepted concept of maximal parity violation. The purpose is essentially didactic since we have no reason to believe that the model of maximal  $CP$  violation discussed bears any relation to reality.

The discussion will be limited to the single process  $\pi \rightarrow l + \bar{\nu}$ , the decay of a spin-zero meson into two spin- $\frac{1}{2}$  leptons, and the cor-

responding antiparticle decay  $\bar{\pi} \rightarrow l + \bar{\nu}$ . In the helicity representation [2] the final state in  $\pi$  decay must be a linear combination of the states  $|RR\rangle$  and  $|LL\rangle$ . The corresponding states in  $\bar{\pi}$  decay will be denoted  $|\bar{R}\bar{R}\rangle$  and  $|\bar{L}\bar{L}\rangle$ . The effects of charge conjugation  $C$  and parity  $P$  are given by

$$P|RR\rangle = |LL\rangle \quad (1a)$$

$$C|RR\rangle = |\bar{R}\bar{R}\rangle \quad (1b)$$

$$CP|RR\rangle = |\bar{L}\bar{L}\rangle. \quad (1c)$$

If  $W(AB)$  is the probability of the final state  $|AB\rangle$  then a measure of parity violation which can vary in absolute magnitude from zero to unity is given by

$$\beta = \frac{W(RR) - W(LL)}{W(RR) + W(LL)} \quad (2)$$

provided the meson  $\pi$  represents a single non-degenerate state.

We discuss first the usual concept of maximal parity violation considering the interaction Hamiltonian [3]

$$H_{\text{int}} = g[\pi l(1 + a\gamma_5)_\nu + \pi^* \bar{\nu}(1 - a\gamma_5)l] \quad (3)$$

where  $\pi$  is the meson field operator,  $l$  and  $\bar{\nu}$  are the two lepton field operators,  $l = l^*\gamma_4$ ,  $\gamma_5$  is Hermitian, and  $a$  is real. It follows from Eq. (3) when we write

$$(1 + a\gamma_5) = \frac{1}{2}(1 + a)(1 + \gamma_5) + \frac{1}{2}(1 - a)(1 - \gamma_5)$$

that

$$\beta = \frac{2a}{1 + a^2} \quad (4)$$

provided (i) that one of the two particles  $l$  or  $\bar{\nu}$  can be considered as having zero mass so that  $\frac{1}{2}(1 + \gamma_5)$  and  $\frac{1}{2}(1 - \gamma_5)$  are projection operators for helicity states and (ii) that lowest-order perturbation theory can be used. The maximum magnitude  $|\beta| = 1$  for the  $P$  violating observable then corresponds directly to the choice  $a = \pm 1$  in the Hamiltonian. This is the essential feature of the concept of maximal parity violation. For later purposes it should be noted that it is obviously sufficient to measure the helicity of either  $l$  or  $\bar{\nu}$  to determine

$\beta$  and that the possibility of such a measurement is assumed.

The decay of  $\bar{\pi}$  can be calculated similarly from the second term in Eq. (3). It suffices for present purposes, however, to apply directly the *CPT* theorem since we have no final state interactions. Since helicities do not change under *T* we find that independent of our choice of  $H_{\text{int}}$

$$W(\bar{L}\bar{L}) = W(RR) \quad (5a)$$

$$W(\bar{R}\bar{R}) = W(LL). \quad (5b)$$

Thus if  $a = +1$  we have  $\bar{\pi}$  decay yielding only left-handed  $l$  and  $\nu$  corresponding to the  $\pi$  decay yielding right-handed  $l$  and  $\bar{\nu}$ . Therefore noting Eqs. (1b) and (1c) we see that there is a maximum violation of *C* invariance but no violation of *CP* invariance. Indeed it is clear from the *CPT* theorem that the observation of *CP* violation must be related to the observation of a *T* violating effect.

To obtain maximal *T* and *CP* violation we replace Eq. (3) by

$$H_{\text{int}} = g[\pi l(1 + i\gamma_5)\nu + \pi^* \bar{\nu}(1 + i\gamma_5)l]. \quad (6)$$

The final state for  $\pi$  decay is now given by

$$|\Psi_1\rangle = (|RR\rangle - i|LL\rangle)/\sqrt{2}$$

and for  $\bar{\pi}$  decay by

$$|\bar{\Psi}_1\rangle = (|\bar{R}\bar{R}\rangle - i|\bar{L}\bar{L}\rangle)/\sqrt{2}.$$

If we consider as basic states  $(\Psi_1, \Psi_2)$  where  $|\Psi_2\rangle = (|RR\rangle + i|LL\rangle)/\sqrt{2}$ , and similarly  $(\bar{\Psi}_1, \bar{\Psi}_2)$  we have from Eqs. (1)

$$P|\Psi_1\rangle = -i|\Psi_2\rangle$$

$$C|\Psi_1\rangle = |\bar{\Psi}_1\rangle$$

$$CP|\Psi_1\rangle = -i|\bar{\Psi}_2\rangle.$$

Thus if we choose as a measure of *P* violation

$$\beta' = \frac{W(\Psi_1) - W(\Psi_2)}{W(\Psi_1) + W(\Psi_2)} \quad (7a)$$

we have  $\beta' = 1$  and maximal *P* violation. Similarly we have maximal



CP violation measured by

$$\gamma = \frac{W(\Psi_1) - W(\bar{\Psi}_2)}{W(\Psi_1) + W(\bar{\Psi}_2)} \quad (7b)$$

since  $\gamma = 1$  for this interaction. On the other hand we clearly have C invariance.

The last paragraph, while formally sufficient, leaves us rather cold. We wish to study in somewhat more detail the nature of the spin correlations in this example and to understand the  $T$  violation that occurs. For this purpose we first consider the possible observables for the decay  $\pi \rightarrow l + \bar{\nu}$  independent of any assumptions about  $H_{\text{int}}$ . This is most easily done by looking at the density matrix  $\rho$  in the composite  $4 \times 4$  spin space of  $l$  and  $\bar{\nu}$ . This may be written [4]

$$\rho = \sum_j a_j S_j. \quad (8a)$$

where the  $S_j$  form a complete set of 16 Hermitian base matrices usually chosen to satisfy the orthonormality relation

$$\text{Tr } S_j S_k = 4\delta_{jk} \quad (8b)$$

Then the possible observables are

$$\langle S_j \rangle = \text{Tr } (\rho S_j) / \text{Tr } \rho \quad (8c)$$

which is proportional to  $a_j$ . Ignoring the unit matrix there are conceivably 15 observables. Our base matrices may be chosen as direct products of the matrices  $(1, \sigma_x, \sigma_y, \sigma_z)$  for  $l$  and  $(1', \sigma'_x, \sigma'_y, \sigma'_z)$  for  $\bar{\nu}$ . We choose the  $z$  axis along the direction of  $l$ , which is opposite to the direction of  $\bar{\nu}$ . In expressing the density matrix in terms of these we make use of relations such as

$$\begin{aligned} |RR\rangle\langle RR| &= \frac{1}{4}(1 + \sigma_z)(1 - \sigma'_z) \\ |RR\rangle\langle LL| &= -\frac{1}{4}(\sigma_x + i\sigma_y)(\sigma'_x - i\sigma'_y). \end{aligned}$$

Because of the axial symmetry of the final state there are only five observables with non-vanishing expectation values. These are listed in the Table and expressed in terms of  $\mathbf{k}$ , the unit vector along the  $z$  axis [5]. By inspection each of these observables may be listed as

conserving or violating  $P$  or  $T$ . The property under  $C$  is then determined by  $CPT$  invariance.

These considerations are now applied to the more general interaction

$$H_{\text{int}} = g[\pi l(\cos \theta + \sin \theta e^{i\alpha}\gamma_5)v + \pi^* \bar{v}(\cos \theta - \sin \theta e^{-i\alpha}\gamma_5)l] \quad (9)$$

of which Eqs. (3) and (6) are special cases. A straightforward calculation gives the density matrix of the final state in  $\pi$  decay

$$\rho = \frac{1}{4}\{1 - \sigma' \cdot \sigma \cos 2\theta - 2\sigma' \cdot k \sigma \cdot k \sin^2 \theta + (\sigma \cdot k - \sigma' \cdot k) \sin 2\theta \cos \alpha + (k \cdot \sigma' \times \sigma) \sin 2\theta \sin \alpha\}. \quad (10)$$

For  $\pi$  decay with the  $z$  axis along the direction of  $l$ ,  $\sigma$  representing  $l$ , and  $\sigma'$  representing  $v$ , we obtain the same result except for a reversal of the signs of the third and fourth terms in accordance with the Table. One measure of parity violation is given by

$$\langle \sigma \cdot k \rangle = \sin 2\theta \cos \alpha. \quad (11)$$

This is indeed identical with the parameter  $\beta$  of Eq. (2), and Eq. (11) agrees with Eq. (4) for the choice  $\alpha = 0$  and the substitution  $a = \tan \theta$ . For maximal parity violation as measured by this parameter we must have  $\alpha = 0$  and  $\theta = \pm \frac{1}{4}\pi$ , which just makes Eq. (9) equivalent to Eq. (3) with  $a = \pm 1$ . Similarly we can choose as a measure of  $CP$  violation

$$\langle \frac{1}{2}k \cdot (\sigma' \times \sigma) \rangle = \sin 2\theta \sin \alpha. \quad (12)$$

This may be shown to be identical with the parameter  $\gamma$  of Eq. (7b). For maximal  $CP$  violation we must have  $\alpha = \frac{1}{2}\pi$  and  $\theta = \pm \frac{1}{4}\pi$  which makes Eq. (9) equivalent to Eq. (6) (or to Eq. (6) with  $i$  replaced by  $-i$ ). *This example puts maximal  $CP$  violation on the same footing as the maximal  $P$  violation discussed above.* Of course, Eq. (12) gives a measure of  $P$  violation as well as of  $CP$  violation, corresponding to the parameter  $\beta'$  of Eq. (7a) rather than  $\beta$ . Requiring either  $|\beta|$  or  $|\beta'|$  to equal unity is a sufficient condition for maximal  $P$  violation but neither is a necessary one. We may say that  $\beta$  measures  $P$  violation associated with  $T$  conservation while  $\beta'$  measures  $P$  violation associated with  $C$  conservation. On the other hand we have found for the process  $\pi \rightarrow l + \bar{\nu}$  there is only one  $CP$  violating observable and that this is

associated with  $C$  and  $PT$  conservation. It is possible in general to have  $CP$  violation associated instead with  $P$  and  $CT$  conservation but not for the process discussed in this note.

TABLE

Observables for  $\pi \rightarrow l + \nu$  which correspond to conservation (c) or violation (v) of parity  $P$ , time-reversal  $T$ , or charge conjugation  $C$ .

	$P$	$T$	$C$	$CP$
(A) $\sigma'_x \sigma_x + \sigma'_y \sigma_y + \sigma'_z \sigma_z = \boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}$	c	c	c	c
(B) $\sigma'_z \sigma_z = \boldsymbol{\sigma}' \cdot \mathbf{k} \boldsymbol{\sigma} \cdot \mathbf{k}$	c	c	c	c
(C) $\sigma_z = \boldsymbol{\sigma} \cdot \mathbf{k}$	v	c	v	c
(D) $\sigma'_z = \boldsymbol{\sigma}' \cdot \mathbf{k}$	v	c	v	c
(E) $\frac{1}{2}(\sigma'_x \sigma_y - \sigma'_y \sigma_x) = \frac{1}{2} \mathbf{k} \cdot (\boldsymbol{\sigma}' \times \boldsymbol{\sigma})$	v	v	c	v

In words the observable (E) given by Eq. (12) corresponds to  $\bar{\nu}$  and  $l$  having transverse polarizations at right angles to each other defining a screw in the direction of motion of  $l$ .  $CP$  invariance would require that in the decay of  $\bar{\pi}$ , the  $\nu$  and  $l$  spins form a screw in the opposite sense to the direction of motion of  $l$ . In fact in the decay of  $\bar{\pi}$  as discussed above the sense of the screw is in the direction of motion of  $l$ . It is thus in this somewhat complicated way that  $CP$  violation manifests itself. It is important to note that it is assumed that the transverse polarizations of  $\bar{l}$  and  $\nu$  are *both* observable. This would not be the case if one of the particles were a zero-mass neutrino the interactions of which were given entirely by a single current of the form of Eq. (9) [6].

The present discussion, in particular Eq. (10), parallels closely the consideration by Bernstein and Michel [7] of possible  $T$  violation in the decay  $\pi^0 \rightarrow \gamma + \gamma$ . This is to be expected because of the formal equivalence between the Stokes parameters used to describe photon polarization and the  $\langle \sigma \rangle$  used to describe spin- $\frac{1}{2}$  particle polarization. It is important in comparing the two cases to realize that it is the appropriate Stokes parameter and not the linear polarization of the photon that is the direct analogue of the "transverse polarization" of  $\bar{\nu}$  and  $l$  discussed above.

To what extent might the maximal  $CP$  violation discussed here be related to reality? Observations of  $\pi$  decay,  $\mu$  decay, and  $\beta$  decay lead

us to believe that leptons and neutrinos are primarily coupled in weak interactions in the  $CP$  invariant manner of Eq. (3) with  $a = 1$ . One cannot rule out a new class of interactions with a coupling weaker by a factor of at least 10 to 100 in which leptons and neutrinos are coupled as in Eq. (6). Suggestions of this sort have been discussed by Hiida [6] and Lotsoff [8] among others. It is also possible to imagine a neutral lepton current of the form (6) in which  $\nu \equiv l$ . Evidence on strange particle decays and elastic neutrino scattering, however, seems to indicate that if weak neutral lepton currents exist they are coupled much more weakly than the usual charged currents.

In submitting this contribution to a collection of papers in honor of Professor Weisskopf, I wish to express my appreciation of many friendly and fruitful contacts with him and to express my gratitude to CERN for the opportunity of spending the years 1957–58 and 1964–65 in its Theoretical Study Division.

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Our phase convention is such that  $|RR\rangle + |LL\rangle$  represents the even parity state for total angular momentum  $J = 0$ . Note that if  $\nu \equiv l$ , in which case we consider the state of  $l + \bar{l}$ , this is the usual convention.
- 3) In the usual theory with vector weak interactions we would write
 
$$H_{\text{int}} = -if \left[ \frac{\partial \pi}{\partial x_\mu} \bar{l} \gamma_\mu (1 + \gamma_5) \nu - \frac{\partial \pi^*}{\partial x_\mu} \bar{\nu} \gamma_\mu (1 + \gamma_5) l \right]$$
 where we have chosen  $\gamma_4 \gamma_\mu^* \gamma_4 = \gamma_\mu$ . This yields Eq. (1) as the effective Hamiltonian for  $\pi$  decay with  $g = f(m_l - m_\nu)$  and  $a = (m_l + m_\nu)/(m_l - m_\nu)$ .
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- 5) For the sake of simplicity the observables listed in the Table have not been required to satisfy the orthonormality relations (8b). In particular, it should be noted that  $\text{Tr} [\frac{1}{2}(\sigma'_x \sigma_y - \sigma'_y \sigma_x)]^2$  equals 2 instead of 4. However, it may be shown that the maximum possible value of the expectation value  $\langle \frac{1}{2}(\sigma'_x \sigma_y - \sigma'_y \sigma_x) \rangle$  equals unity as is the case for the normalized observables (B), (C) and (D).
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# THE $SU_3$ MASS FORMULA

KERSON HUANG

*Massachusetts Institute of Technology, Cambridge, Massachusetts*

*(Received May 6, 1965)*

The well-known mass formula of Gell-Mann and Okubo [1] for particles belonging to  $SU_3$  super-multiplets has been experimentally verified to good accuracy. In this note we point out that the usual derivation for it is incomplete, and that its validity seems to indicate the existence of a mechanism that regulates the masses of fundamental particles. We suggest that this mechanism might be the so-called "bootstrap".

In the usual derivation of the mass formula, one assumes that the Hamiltonian  $H$  for the fundamental particles is of the form

$$H = H_0 + H_1, \quad (1)$$

where  $H_0$  is invariant under  $SU_3$ , and  $H_1$  is a small perturbation that transforms under  $SU_3$  like the hypercharge. If  $H_1$  were absent, all particles in an  $SU_3$  super-multiplet would have the same mass. Calculating the mass splitting due to  $H_1$  in first-order perturbation theory leads to the Gell-Mann-Okubo mass formula. The calculation is analogous to that of the Zeeman effect in atomic spectra, though less trivial, because one is concerned here with the group  $SU_3$  instead of  $SU_2$ . In the atomic Zeeman effect, the use of first-order perturbation theory is justified because the system has discrete energy levels, and the energy perturbations are small compared to level spacings. The situation is different in the  $SU_3$  problem, because some of the particles are not discrete states but resonances. As we turn on  $H_1$ , two effects contribute to the mass splitting: a direct effect of  $H_1$  that is similar to the Zeeman effect, and an indirect effect arising from shifts in the thresholds of scattering channels. The usual derivation of the mass formula takes into account only the "Zeeman effect".

For concreteness let us consider the mass formula for the baryon decuplet of spin  $\frac{3}{2}$ , whose members are the isospin multiplets  $\{N_{\frac{3}{2}}^*$ ,

$Y_1^*, \Xi_{\frac{1}{2}}^*, \Omega_0\}$ , where the subscripts refer to the isospin. The Gell-Mann-Okubo mass formula predicts that these isospin multiplets have equally spaced masses, a prediction verified by experiments to within 2 %. We note, however, that with the exception of  $\Omega_0$  these particles are resonances occurring in various channels of meson octet-baryon octet scattering. Let us try to compute their masses in a simple model of resonance scattering, to see whether the mass splitting induced by  $SU_3$  violation is purely a "Zeeman effect", as the usual derivation of the mass formula implies.

The multiplets  $\{N_{\frac{1}{2}}^*, Y_1^*, \Xi_{\frac{1}{2}}^*, \Omega_0\}$  are distinguished by their isospin  $I$  and hypercharge  $Y$ . They are resonances or bound states in all scattering channels having the specific  $I$  and  $Y$ , (and total angular momentum  $J = \frac{3}{2}$ ). As such they are represented by energy poles in the various scattering amplitudes. With the neglect of all but strongly interacting two-body scattering channels, the positions of the decuplet poles in relation to the thresholds of the relevant channels are as shown in Fig. 1, where relevant energies are given in MeV. If  $H_1$  were zero, all the thresholds in Fig. 1 would occur at the same energy, and so would all the decuplet poles. For our purpose it is not important whether in this hypothetical limit the decuplet should be a bound state or a resonance. Whatever the case, it is clear from Fig. 1 that when  $H_1$  is turned on, some decuplet poles must cross some thresholds, so that what might originally have been an open channel becomes a closed channel, or vice versa. The mass spacing of the decuplet is two to three times smaller than the spread of thresholds for given  $(I, Y)$ . We should therefore expect that the splitting of the thresholds has an important influence on the decuplet masses. To calculate the latter we use the simplest theory of many-channel resonance scattering, the  $R$ -matrix theory of Wigner and Eisenbud [2].

The  $S$ -matrix for a given set of quantum numbers  $(I, Y)$  is an  $N \times N$  matrix, where  $N$  is the total number of channels having those quantum numbers. It is to be obtained from the  $R$ -matrix, which is an  $N \times N$  real symmetric matrix that in principle can be obtained by solving an eigenvalue problem involving the Hamiltonian. For the present problem we use the one-level formula

$$R = \frac{\gamma \times \gamma}{\varepsilon - E}, \quad (2)$$

where  $E$  is the total c.m. energy, and  $\gamma$  is an  $N$ -component vector, whose components are real and positive. The number  $\varepsilon$  is an eigenvalue of  $H$  under a specific hermitian boundary condition, which makes all its eigenvalues real and discrete. The  $S$ -matrix is then given by

$$S = \omega \frac{1 + iBRB}{1 - iBRB} \omega, \quad (3)$$

where  $\omega$  and  $B$  are diagonal  $N \times N$  matrices, with

$$\omega_{mn} = \delta_{mn} \exp (ia_n q_n), \quad (4)$$

$$B_{mn} = \delta_{mn} q_n^{\frac{3}{2}},$$

where  $a_n$  is the channel radius of the  $n^{\text{th}}$  channel, and  $q_n$  is the relative momentum in the  $n^{\text{th}}$  channel at total c.m. energy  $E$ . Thus  $q_n$  is real for open channels, pure imaginary for closed channels. Substituting (2) into (3) we obtain

$$S = \omega \frac{\varepsilon - E + i\phi \times \phi}{\varepsilon - E - i\phi \times \phi} \omega, \quad (6)$$

where  $\phi = B\gamma$ . Next we note the identity

$$\frac{1}{1 + c\phi \times \phi} = 1 - \frac{c\phi \times \phi}{1 + c\phi^2}, \quad \phi^2 \equiv \sum_n \phi_n \phi_n, \quad (7)$$

which can be verified by multiplying both sides by  $1 + c\phi \times \phi$ . Using (7) to rewrite the denominator in (6), we find after a few algebraic steps that

$$S = \omega \left[ 1 + \frac{2i\phi \times \phi}{\varepsilon - E - i\phi^2} \right], \quad (8)$$

which is a one-level Breit-Wigner formula\*. It shows that, for given  $(I, Y)$ , all the  $N \times N$   $S$ -matrix elements have a pole at  $E = \varepsilon - i\phi^2$ , which represents a bound state with the quantum numbers  $(I, Y)$  if  $i\phi^2$  is real, a resonance otherwise. The mass of the bound state or resonance is given by

$$m = \varepsilon + Im \sum_n q_n^3 \gamma_n^2, \quad (9)$$

\* Eq. (8) is essentially Eq. (4.26) in Chap. 10 of Blatt and Weisskopf, except that in the latter  $Re(i\phi^2)$  is lumped into  $\varepsilon$ .

where the second term is evaluated at total c.m. energy  $m$ . We see that closed channels, with a pure imaginary  $q_n$ , contribute to the mass, but open channels do not.

Since  $\varepsilon$  is a discrete eigenvalue of  $H$ , we calculate it for the different members of the decuplet by treating  $H_1$  in first order perturbation theory, with the result that  $\varepsilon$  obeys the Gell-Mann-Okubo mass formula, i.e., it is equally spaced for  $\{N_{\frac{3}{2}}^*, Y_1^*, \Xi_{\frac{1}{2}}^*, \Omega_0\}$ . This is the "Zeeman effect" referred to earlier.

The second term in (9) depends on the detailed form of the Hamiltonian, and we know of no general principle dictating that it too must be equally spaced. We may calculate it, however, as follows. To obtain  $q_n$ , we use the observed mass values of the decuplet, the meson octet, and the baryon octet. Further, we can determine one of the  $\gamma_n^2$ , say that for  $N_{\frac{3}{2}}^* \rightarrow \pi N$ , by the experimentally observed width of  $N_{\frac{3}{2}}^*$ . All the other  $\gamma_n^2$  are then proportional to it through  $SU_3$  Clebsch-Gordan coefficients, if we assume  $SU_3$  symmetry for this calculation. The calculations [3] are straightforward and will not be described in detail. The result, in the context of this calculation, is surprising: we find that the second term in (9) is also equally spaced for  $\{N_{\frac{3}{2}}^*, Y_1^*, \Xi_{\frac{1}{2}}^*, \Omega_0\}$ . Numerically this term is an important contribution. For example, for  $N_{\frac{3}{2}}^*$ , it is about twice the decuplet mass spacing.

The same conclusion is arrived at in more sophisticated calculations using a Chew-Low type theory [3], and a relativistic  $N/D$  method [4].

Thus, beginning by questioning the Gell-Mann-Okubo mass formula, we end up verifying it. In the process, however, we have lost the theoretical understanding that we thought we had originally. For it seems to be pure accident that the number of closed channels, and the masses of the octets, are just so arranged as to yield the equal-spacing mass rule. On the other hand, it is hard to believe that the result, so simple and model-independent, can be purely accidental. It seems plausible, therefore, that there is a general mechanism, so far undiscovered, that requires the result. In the calculation described the mechanism must be buried somewhere in the experimental numbers fed in.

A natural candidate for this mechanism seems to be the "bootstrap", the idea behind which is that the masses of all the fundamental particles are related in some self-consistent way. According to this idea, the various thresholds in Fig. 1, which determine the decuplet poles, can-



not occur at arbitrary positions, but are in turn determined in some manner by the decuplet poles. It is conceivable that some such scheme may lead to the equal-spacing rule as a consequence of a self-consistency requirement. A step towards such an explanation is the recent work of Dashen and Frautschi [5], who show how the bootstrap mechanism may explain “octet dominance”, which means that the mass splitting transforms under  $SU_3$  like the hypercharge. However,

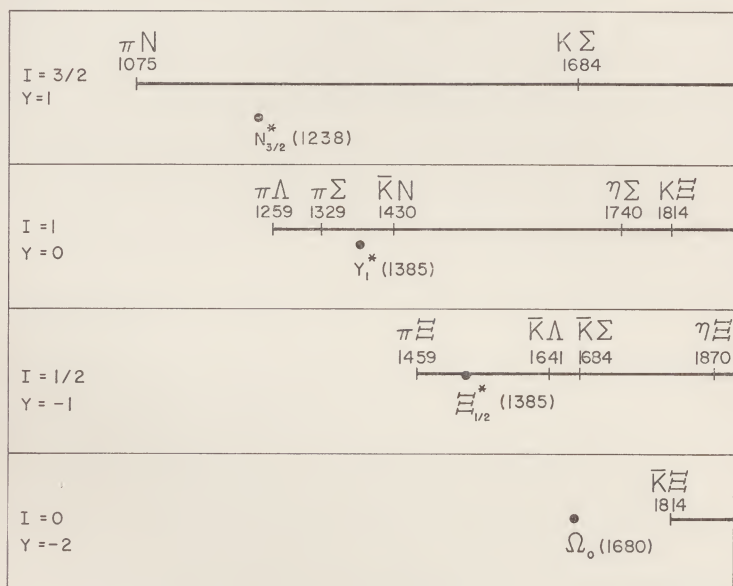


Fig. 1.

they treat the mass splitting as a first order “Zeeman effect”, and do not touch on the threshold effects discussed above. Thus, although no reference is made to a Hamiltonian, the result is equivalent to the statement that  $H_1$  transforms like the hypercharge. A more realistic bootstrap scheme must include the threshold effects. If the “octet dominance” proves to persist even when threshold effects are taken into account, one would have a more satisfactory understanding of the mass formula.

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# ARE WAVE FUNCTIONS FINITE?

F. E. LOW

*Massachusetts Institute of Technology, Cambridge, Massachusetts*

*(Received May 6, 1965)*

We ask the following question: is there a principle in nature which requires that a wave function  $\psi(\mathbf{r}, t)$  which is finite (for all  $\mathbf{r}$ ) at some time  $t_0$  remain finite for all  $t$ ?

It is obvious that in the non-relativistic approximation no such principle exists, since we know that a free particle  $\delta$  function spreads into a finite wave function (and conversely, therefore, a finite wave function can contract into a  $\delta$  function). Relativistic kinematics does not appear to permit this to happen, because of the relation between energy and momentum.

Consider now the center of mass wave function for a two-particle system, with an incident plane (but not monochromatic) wave in the  $z$  direction. We write, neglecting spin, for large separation of the two particles,

$$\psi(\mathbf{r}, t) = \int_{-\infty}^{\infty} dk a(k) e^{-ikz_0} \left[ e^{ikz} + \frac{1}{r} e^{ikr} f(k, \theta) \right] e^{-i\omega(k)t}. \quad (1)$$

The first term in Eq. (1) is the incident wave. We have inserted the factor  $e^{-ikz_0}$  to imply that at  $t = 0$  it is centered near  $z_0$  (a large negative number). The second term is the scattered wave which at  $t = 0$  is presumably zero. The function  $f(k, \theta)$  is the scattering amplitude:

$$|f(k, \theta)|^2 = \frac{d\sigma_{el}}{d\Omega} \quad (2)$$

and  $\text{Im} f(k, 0) = k\sigma_T(k)/4\pi$  where  $d\sigma_{el}/d\Omega$  is the elastic differential cross-section, and  $\sigma_T$  the total cross-section.

Our principle may now be formulated as follows: if  $a(k)$  is such that

$$\int_{-\infty}^{\infty} e^{ik(z-z_0)} a(k) dk = \text{finite (all } z) \quad (3)$$

then

$$\int_0^\infty e^{i[k(r-z_0)-\omega t]} dk a(k) f(k, \theta) = \text{finite (all asymptotic } r, t). \quad (4)$$

Evidently, by choosing  $z = z_0$ ,  $a(k)$  real and positive in Eq. (3) and  $t = r - z_0$  in Eq. (4), we must have, since  $\omega \rightarrow k$  as  $k \rightarrow \infty$ ,  $\text{Im } f(k, 0)$  bounded as  $k \rightarrow \infty$ . It then follows from Eq. (2)

$$k\sigma_T < \infty. \quad (5)$$

The present trend of high energy experiments appears to contradict our conclusion, and thus invalidate our suggested principle. Of course, the mathematical formulation we have given is at best somewhat dubious. Nevertheless, we believe the original principle must be abandoned.

One might ask whether the principle might hold only for physically realizable states, possessing normalizable wave-functions. In the latter case our example would be ruled out so that the bound on  $f(k, 0)$  might be expected to be less stringent.

Using a similar argument to those given above one finds in this case the bound

$$\lim_{k \rightarrow \infty} \int_0^{t_1} f(k, t) dt < \infty \quad (6)$$

where  $t = 2k^2(1 - \cos \theta)$ , and  $t_1$  is a fixed momentum transfer. Eq. (6) is also in disagreement with experiment.



# AN ELEMENTARY NOTE ABOUT "MIXTURES"

B. D'ESPAGNAT

*University of Paris*

(Received May 7, 1965)

## 1. INTRODUCTION

Let  $x_1, \dots, x_p$  be specified coordinates and let us consider an ensemble  $E$  of systems  $S$  depending on these variables. If all these systems have the same wave function  $u$ ,  $E$  is called a *pure case*. If  $E$  is made of subensembles  $E_1, E_2, \dots, E_r$  which are pure cases with wave functions  $u_1, \dots, u_r$ ,  $E$  is called a mixture (we assume that  $u_1, \dots, u_r$  are different from each other).

The word "mixture" is also used in a somewhat different context. Let  $x_{p+1}, \dots, x_n$  be other, specified coordinates. Let  $A$  be a system that depends on these coordinates and let us consider the interaction of a system  $S$  with a system  $A$ . It is convenient for that purpose to introduce the "larger" system  $\Sigma = S + A$  and to consider an ensemble of systems  $\Sigma$ . If, initially, (i.e. before the interaction) both the constituent ensembles of  $S$  systems and of  $A$  systems are pure cases, described by  $u_0$  and  $v_0$  respectively, then obviously the ensemble of  $\Sigma$  is also a pure case, corresponding to the product  $u_0 \cdot v_0$ . After the interaction of  $S$  with  $A$ , that same ensemble is still a pure case, which is deduced from  $u_0 \cdot v_0$  by applying to it the time-dependent Schrödinger equation. If, however, one considers separately the ensemble of systems  $S$  and the ensemble of systems  $A$  after the interaction has taken place, one immediately sees that, in general, neither of them is a pure case. One then usually says that they are a mixture.

The purpose of the present note is to stress that these two acceptations of the word "mixture" do not really describe the same kind of situation and that therefore our language should distinguish them.

## 2. A SIMPLE EXAMPLE

Following Bohm [1], we shall first consider a very simple example of a situation where two systems have interacted in the past. Let us imagine

that a spin zero molecule decays into two spin  $\frac{1}{2}$  objects  $S$  and  $A$  with conservation of the total *spin* (we consider of course an ensemble of such molecules). After the decay has taken place the ensemble of large systems  $\Sigma = S + A$  is a pure case described by the spin wave function:

$$\psi = (u_+ v_- - u_- v_+)/\sqrt{2} \quad (1)$$

where  $u_{\pm}$  and  $v_{\pm}$  are the eigenfunctions of the spin component  $\sigma_z$  along the third axis of  $S$  and  $A$  respectively. From expression (1), statistical predictions concerning results of observations are easily derived. In particular one finds that (in accordance with intuition):

a) the probability (statistical frequency) for observing  $\sigma_z(S) = +\frac{1}{2}$  is  $\frac{1}{2}$

the probability (statistical frequency) for observing  $\sigma_z(S) = -\frac{1}{2}$  is  $\frac{1}{2}$

b) the probability (statistical frequency) for observing  $\sigma_z(S) = \frac{1}{2}$  and  $\sigma_z(A) = -\frac{1}{2}$  is  $\frac{1}{2}$ .

c) the probability (statistical frequency) for observing  $\sigma_z(S) = \frac{1}{2}$  and  $\sigma_z(A) = \frac{1}{2}$  is 0.

d) exactly the same results hold also with  $\sigma_z$  replaced by  $\sigma_x$ .

If we focus for a moment our attention on the predictions above that bear on system  $S$  alone, we immediately see that these same predictions would also hold if instead of (1) the mixture:

$$\begin{cases} u_+ & \text{for one half of all the systems } S \\ u_- & \text{for the other half} \end{cases} \quad (2)$$

had been considered.

This is a mixture in the first sense given in the introduction i.e. it splits into two subensembles  $E_+$ ,  $E_-$  (corresponding to  $u_+$  and  $u_-$  respectively) that are pure cases\*. Let us now ask the question: is it possible to describe the systems  $\Sigma$  of the ensemble  $\mathcal{E}$  in such a way that:  $\alpha$ ) all the predictions a), b), c), d) hold,  $\beta$ ) each system  $S$  is either in  $E_+$  or in  $E_-$ ? Now, if a system  $S$  is in  $E_+$ , a measurement of its  $\sigma_z$  is predicted to give  $+\frac{1}{2}$  and therefore, through c), a measurement of  $\sigma_z$  on the system  $A$  which is associated to  $S$  is predicted to give with certainty  $\sigma_z(A) = -\frac{1}{2}$ ; in other words the system  $\Sigma = S + A$  is part

\* Such mixtures could be called "proper" mixtures.

of the ensemble (pure case) described by:

$$u_+ v_-.$$

Similarly if a system  $S$  is in  $E_-$ ,  $\Sigma$  is part of the ensemble described by

$$u_- v_+.$$

Thus, if the ensemble of systems  $S$  is in a mixture in the first sense, namely (2), the ensemble of systems  $\Sigma$  is also in a mixture in the first sense, namely

$$\begin{cases} u_+ v_- & \text{for one half of all the systems } \Sigma \\ u_- v_+ & \text{for the other half.} \end{cases} \quad (3)$$

Now the predictions a), b), c) of (1) are also predictions of (3). Some of the predictions d) however, those that bear on correlations, are incompatible with (3). For instance (3) predicts that the probability of finding  $\sigma_x(S) = +\frac{1}{2}$  and  $\sigma_x(A) = +\frac{1}{2}$  is  $\frac{1}{4}$  whereas (1) predicts that this same probability is zero.

If the systems  $S$  were objectively well described by a mixture in the first sense or "proper" (i.e. if half of them really *had*  $\sigma_z = +\frac{1}{2}$  and half of them  $\sigma_z = -\frac{1}{2}$ ) all the observable predictions that one can derive from this description by using the usual rules of quantum mechanics should obviously be correct. This, as we have seen, is not the case. It is therefore not correct to say that each system  $S$  has an objective reality of its own which is an individual element of an ensemble described by (2) (nor of course by any other mixture in the first sense). The ensemble of systems  $S$  is on the other hand correctly described as a mixture in the second sense by definition. As a consequence the concepts of mixture in the first sense and of mixture in the second sense (or "improper" mixture) are not identical with each other.

### 3. THE GENERAL CASE

The general case can be treated in exactly the same way as the example above. Let  $S$  and  $A$  be two systems that have interacted in the past. Von Neumann has shown that it is always possible to find two systems of orthonormal wave functions,  $u_k$  and  $v_k$  of the variables of  $S$  and  $A$

respectively such that the wave function of  $\Sigma = S + A$  takes the form:

$$\psi = \sum_k c_k u_k v_k. \quad (4)$$

Let  $U$  and  $V$  be the corresponding observables. The only ensemble in the first sense that reproduces all the predictions of (4) as regards possible measurements on  $S$  alone, possible measurements on  $A$  alone and correlations between  $U$  and  $V$  measurements is

$$\begin{cases} \dots \\ u_k v_k \text{ for a fraction } |c_k|^2 \text{ of all systems } \Sigma \\ \dots \end{cases} \quad (5)$$

As shown by Furry [2], this mixture does not however reproduce the prediction of (4) as regards the correlation between measurements on one variable of  $S$  other than  $U$  and one variable of  $A$  other than  $V$ . Using these results the extension of the argument in section 2 is trivial.

We may point out at this stage that if one wants to express the difference between the two meanings of the word mixture in terms not of ensembles but of properties of individual systems one has simply to state again the rather well-known fact that if  $S$  and  $A$  have interacted in the past and have now ceased to interact there is no state vector – known or unknown – that correctly describes the “physical reality” of  $S$  alone, whatever this expression means. The reason for that is again that, if such a picture were a true description of reality as it really is, all its consequences, and not only those pertaining to system  $S$ , should be correct, which is not the case.

#### 4. THE DENSITY MATRIX FORMALISM

A) Instead of describing a pure case by its state vector  $|u\rangle$  one can also describe it by means of the so called “statistical operator”

$$M = |u\rangle\langle u| \quad (6)$$

or by the corresponding “density matrix”:

$$M_{m,n} = \langle m|M|n\rangle \quad (7)$$

which satisfies

$$\text{Tr } M = 1 \quad (8)$$

and

$$M^2 = M. \quad (9)$$



The mean value  $\overline{\mathcal{R}}$  of an observable  $\mathcal{R}$  described by the hermitian operator  $R$  is, as immediately seen:

$$\overline{\mathcal{R}} = \text{Tr} [MR]. \quad (10)$$

B) Similarly let us consider a mixture in the first sense, made of systems in states  $|u_i\rangle$  with relative abundancy  $w_i$  ( $\sum_i w_i = 1$ ). This mixture can also be described by means of the statistical operator:

$$M' = \sum_i w_i |u_i\rangle\langle u_i| \quad (11)$$

or by the corresponding density matrix

$$M'_{mn} = \langle m|M'|n\rangle \quad (12)$$

which satisfies

$$\text{Tr} M' = 1. \quad (13)$$

The mean value  $\overline{\mathcal{R}}$  of an observable  $\mathcal{R}$  in the ensemble is again given by:

$$\overline{\mathcal{R}} = \sum_i w_i \langle u_i|R|u_i\rangle = \text{Tr} [M'R]. \quad (14)$$

Thus the density matrix formalism provides a compact expression for interesting physical quantities such as mean values, even in those cases where the systems are in a mixture in the first sense. The only difference with the pure case is that now

$$M'^2 \neq M'. \quad (15)$$

C) Let us now consider a mixture in the second sense. For that purpose we expand the wave function of the large system  $\Sigma$  considered above in terms of the eigenfunctions  $u'_m$  of any variable  $U'$  of  $S$  as:

$$\psi(x_1 \dots x_n) = \sum_m u'_m(x_1 \dots x_p) A_m(x_{p+1} \dots x_n) \quad (16)$$

where the  $A_m$  are coefficients, depending of course on the other variables in  $\Sigma$ . We then expand  $A_m$  in terms of the eigenfunctions  $v'_n$  of a variable  $V'$  of  $A$  and carry the result into (16). This gives

$$\psi(x_1 \dots x_n) = \sum_{mn} C_{mn} u'_m v'_n \quad (17)$$

or in terms of state vectors

$$|\psi\rangle = \sum_{mn} C_{mn} |u'_m\rangle |v'_n\rangle. \quad (18)$$

$|\psi\rangle$  is the state vector of the ensemble  $\mathcal{E}$  (pure case) of large systems  $\Sigma$ . Let now  $\mathcal{R}$  be an observable pertaining to  $S$ . Its mean value on  $\mathcal{E}$  is:

$$\begin{aligned}\overline{\mathcal{R}} &= \langle \psi | \mathcal{R} | \psi \rangle = \sum_{mnr} C_{mn}^* C_{rs} \langle u'_m | \mathcal{R} | u'_r \rangle \langle v'_n | v'_s \rangle \\ &= \sum_{mnr} C_{mn}^* C_{rn} \langle u'_m | \mathcal{R} | u'_r \rangle \\ &= \text{Tr} [M' R]\end{aligned}\quad (19)$$

with (using this special basis for  $R$ )

$$M'_{rm} = \sum_j C_{rj} C_{mj}^* \quad (20)$$

The conclusion is that also in the case of a mixture in the second sense – the mixture of subsystems  $S$  – formula (13) holds, provided that  $M'$  is defined by Eq. (20), which is independent of  $\mathcal{R}$ .  $M'$  thus defined satisfies moreover (13) and (15). This is in fact the reason why the “mixtures in the first sense” and the “mixtures in the second sense” have both received the same name “mixtures”.

## 5. CONCLUSION

Let, again,  $S$  be a subsystem of a larger system  $\Sigma = S + A$  and let  $\mathcal{E}$  be an ensemble of  $\Sigma$ . As has been recalled in section 4, as long as we consider only future measurements on quantities pertaining to systems  $S$  alone, the ensemble of systems  $S$  can be viewed as a “mixture in the first sense”, i.e. as composed of subensembles that are pure cases. This does not mean however that the ensemble of systems  $S$  is *physically identical* to a mixture in the first sense for, if this statement were true, *all* its observable consequences should of course be correct and not only those pertaining to measurements on systems  $S$  alone. This, we know, is not the case.

In the elementary description of the theory of measurement it is sometimes said that when the state vector of the corresponding system  $\Sigma$  is (4) the corresponding ensemble of systems  $S$  is a “mixture” and that *therefore* a particular system  $S$  has *either*  $u_1$  *or*  $u_2$  *or* ...  $u_3$  ... *or*  $u_k$  ... for its wave function, the corresponding wave function for  $A$  being of course the  $v$  with the corresponding index. This then is used as an argument for showing that a measurement of  $U$ , using  $A$  as an instrument, induces no physical change on  $S$  and represents simply an increase of our knowledge (for, it is said, it is just ascertaining that the

wave function of  $S$  is one of the particular  $u_k$ , which it was already before). The fallacy of this argumentation is due to the fact that it uses the word "mixture" in the two different senses. From the fact that the ensemble of systems  $S$  is a "mixture in the second sense" there is of course no reason to conclude that it is a "mixture in the first sense", and that therefore its constituent systems are in one or other of the states described by the different  $u_k$ . In fact the true conclusion to be drawn is exactly the opposite.

All that is said above is elementary and, undoubtedly, generally known. The only point we want to stress is that, since the two kinds of "mixture" are really different concepts, it would be both convenient and appropriate to distinguish them in the language. This would make a description of the real problems involved in a theory of measurement more transparent and would therefore be a suitable approach to the various efforts\* that have been made at solving them.

\* For a bibliography on this subject see for instance, ref. [3].

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# BOSON BETA DECAY

D. C. PEASLEE

*Australian National University, Canberra, Australia*

*(Received May 7, 1965)*

The following note considers leptonic decay of bosons in the closest possible analogy with ordinary beta decay. Each boson is represented as a sum over "nuclei" composed of  $N$  baryons and  $N$  antibaryons. Conserved vector current (CVC) for  $\Delta S = 0$  is then just the well known rule that the Fermi matrix element for superallowed nuclear transitions is unity.

If one repeats this approach for  $\Delta S \neq 0$  transitions, no CVC theorem appears; nor is it possible to make general statements about axial vector transitions except within supermultiplets. In this connection, the interest is noted in measuring  $K \rightarrow e + \nu + 2\pi$  decays.

## 1. BARYON-ANTIBARYON MODEL

Represent every boson as a sum over baryon-antibaryon states.

$$\begin{aligned}\phi &= a_2(\bar{\psi}\psi) + a_4(\bar{\psi}\bar{\psi}\psi\psi) + \dots \\ \sum |a_{2N}|^2 &= 1, \quad a_0 \equiv 0.\end{aligned}\tag{1}$$

For present purposes we can regard the antibaryons as real particles, so that Eq. (1) is an expansion in states of 2, 4, ... baryons. Each such state can be approached in exact analogy to a 2-, 4-, ... body nucleus in low energy nuclear physics. Even the inconstancy of baryon number  $2N$  is already encountered in calculations on the collective model, although in that case  $(\Delta N/N)$  is generally small.

On this basis, it is easy to "derive" the conserved vector current hypothesis as equivalent to the superallowed beta decay of nuclei. Consider first the non-relativistic approximation: Fermi beta decay operators for the successive terms in Eq. (1) are [1]

$$F_{N, \pm} = \sum_{n=1}^{2N} (1)_n \tau_{n, \pm}.$$



If two boson states  $\phi(I, I_z)$  and  $\phi'(I, I_z \pm 1)$  are members of an isotopic multiplet, the Fermi matrix element between them is

$$\begin{aligned}\langle \phi' | F_{\pm} | \phi \rangle &= \sum_{n=1}^{\infty} \langle \phi' | F_{N, \pm} | \phi \rangle = \sum_{n=1}^{\infty} |a_n|^2 \langle T_{\pm} | \rangle \\ &= \langle T_{\pm} | \rangle\end{aligned}\quad (2)$$

where the relation [2]

$$\langle | \sum_{n=1}^{2N} \tau_{n, \pm} | \rangle = \langle T_{\pm} | \rangle \quad (3)$$

holds independent of  $N$ .

Equation (2) is the same as for a single baryon and is essentially the CVC hypothesis [3]. It only remains to supply the relativistic form: If one writes  $\{\gamma_{4n}, \gamma_n\}$  in the expressions for  $F_{2N}$  and takes the non-relativistic limit under  $E' \approx E$ , the matrix element becomes

$$\begin{aligned}\{1, i(\mathbf{p} + \mathbf{p}')/2E\} &\approx [\sqrt{4EE'}]^{-1} \{(E + E'), i(\mathbf{p} + \mathbf{p}')\} \\ &= (\partial_{\mu} \phi'^* \phi - \phi'^* \partial_{\mu} \phi) / \sqrt{4EE'}\end{aligned}$$

in a covariant field theory. The complete interaction form is then

$$g \{ \partial_{\mu} \phi'^* T_{\pm} \phi - \phi'^* T_{\pm} \partial_{\mu} \phi \} l_{\mu}^{\pm} \quad (4)$$

where  $l_{\mu}$  is the lepton current. There are no approximations here, and the only obvious source of failure in practice for Eq. (4) is the usual isotopic spin impurity of  $\phi$  induced by Coulomb interaction.

The argument above holds for all vector currents with  $\Delta S = 0$ ; for strangeness-changing transitions the Fermi operator involves

$$\sum_{n=1}^{2N} (\Delta \tau = \frac{1}{2})_{n, \pm \frac{1}{2}}$$

for which no theorem like Eq. (3) exists. The weak interaction itself therefore seems to preclude a CVC theorem for  $\Delta S = \pm 1$ , or at least a derivation along the present lines; this conclusion is independent of any symmetry features of the strong interactions.

What about the Gamow-Teller matrix elements? When  $\Delta S = 0$  they can in principle be specified independent of  $N$  for decays within the same nuclear supermultiplet [4]. This was also true for the Fermi matrix elements; the difference is that while  $\langle F \rangle$  depends only on  $T_{\pm}$

and is otherwise the same for all supermultiplets, the value of  $\langle G-T \rangle$  varies with the supermultiplet. No statement so universal as CVC can thus be made. Unfortunately, the most obvious boson supermultiplet transitions seem likely to have  $\langle G-T \rangle = 0$ : *e.g.*,  $\eta \leftrightarrow \rho^\pm$ ,  $\phi \leftrightarrow \pi^\pm$ , and  $\omega \leftrightarrow \pi^\pm$ .

Gamow-Teller transitions with  $\Delta S \neq 0$  suffer from a combination of the above difficulties, and no predictions can be made on the above basis. Beta decay transitions  $\pi^\pm$ ,  $K^\pm \rightarrow (\text{vacuum})$  depend on the  $G-T$  interaction but correspond to first forbidden transitions involving orbital coordinates. The apparent success of a scheme [5] for using  $SU_3$  symmetry to extend the general notion of CVC to all kinds of leptonic decay seems all the more remarkable in the light of these comments.

## 2. FERMI MATRIX ELEMENTS FOR $\Delta S = 0$

We calculate some examples of  $F_1$  as a concrete illustration of the preceding remarks. Write  $a_{2N} = 0$  for  $N > 1$ , so that

$$\begin{aligned} \pi^+ &= \frac{1}{2} \left[ \begin{aligned} &a\{\bar{n}(1)p(2) + \Xi^0(1)\bar{\Xi}^-(2)\} \\ &+ b\{\bar{\Sigma}^-(1)\Lambda(2) + \Sigma^+(1)\bar{\Lambda}(2)\} + (1 \leftrightarrow 2) \\ &+ c\{\bar{\Sigma}^-(1)\Sigma^0(2) - \Sigma^+(1)\bar{\Sigma}^0(2)\} \end{aligned} \right] {}^1\chi_0 \\ \pi^0 &= \frac{1}{2} \left[ \begin{aligned} &a/\sqrt{2}\{\bar{n}(1)n(2) - \bar{p}(1)p(2) + \Xi^-(1)\bar{\Xi}^-(2) - \Xi^0(1)\bar{\Xi}^0(2)\} \\ &+ b\{\bar{\Sigma}^0(1)\Lambda(2) + \Sigma^0(1)\bar{\Lambda}(2)\} + (1 \leftrightarrow 2) \\ &+ c\{\Sigma^+(1)\bar{\Sigma}^+(2) - \bar{\Sigma}^-(1)\Sigma^-(2)\} \end{aligned} \right] {}^1\chi_0 \\ \pi^- &= \frac{1}{2} \left[ \begin{aligned} &-a\{\bar{p}(1)n(2) + \Xi^-(1)\bar{\Xi}^0(2)\} \\ &-b\{\bar{\Sigma}^+(1)\Lambda(2) + \Sigma^-(1)\bar{\Lambda}(2)\} + (1 \leftrightarrow 2) \\ &+ c\{\Sigma^0(1)\bar{\Sigma}^+(2) - \bar{\Sigma}^0(1)\Sigma^-(2)\} \end{aligned} \right] {}^1\chi_0 \end{aligned} \quad (5)$$

where  ${}^1\chi_0$  is a scalar function (symmetric antisymmetric) in (space, spin) coordinates of baryon and antibaryon. It is normalized to unity and hence  $(a^2 + b^2 + c^2) = 1$ , the phases all being real by charge conjugation invariance: the relative signs of the  $(1 \leftrightarrow 2)$  terms are fixed to satisfy the Pauli principle. Here  $\bar{n}$  means antineutron, etc. and the relative signs within the  $\{ \}$  are chosen to make  $A\pi = -\pi$  under a standard convention [6].

Note that one-to-one correspondence does not obtain between the sign of  $A$  and the choice of  $f$  or  $d$  couplings in  $SU_3$ .

If the Fermi coupling for  $p \rightarrow n$  is  $V_a$ , then  $AP$  invariance requires it to be  $V_a$  for  $\Xi^0 \rightarrow \Xi^-$  as well; of course it is  $-V_a$  for  $\bar{n} \rightarrow \bar{p}$  and  $\bar{\Xi}^- \rightarrow \bar{\Xi}^0$ . In a similar way we complete the tableau

$$\begin{aligned} V_a: & (p \rightarrow n, \Xi^0 \rightarrow \Xi^-), -(\bar{n} \rightarrow \bar{p}, \bar{\Xi}^- \rightarrow \bar{\Xi}^0) \\ \sqrt{2}V_b: & (\Sigma^+ \rightarrow \Lambda, \Lambda \rightarrow \Sigma^-), -(\bar{\Lambda} \rightarrow \bar{\Sigma}^+, \bar{\Sigma}^- \rightarrow \bar{\Lambda}) \\ \sqrt{2}V_c: & (\Sigma^+ \rightarrow \Sigma^0, \bar{\Sigma}^- \rightarrow \bar{\Sigma}^0), -(\Sigma^0 \rightarrow \Sigma^-, \bar{\Sigma}^0 \rightarrow \bar{\Sigma}^+) \end{aligned} \quad (6)$$

The non-relativistic matrix element follows at once:

$$\begin{aligned} \langle \pi^0 | F_- | \pi^+ \rangle &= \langle \pi^- | F_- | \pi^0 \rangle \\ &= \sqrt{2}V_a[1 - 2bc(V_b/V_a) - (b^2 + c^2)(1 - V_c/V_a)] \end{aligned} \quad (7)$$

This can be independent of  $b$  and  $c$  only if

$$V_b = 0, V_c = V_a \quad (8)$$

It is therefore not sufficient just to say loosely that  $\langle F \rangle$  is proportional to isotopic spin; for when  $Y = 0$  there are two possible isotopic spins in the octets:  $T(1 \leftrightarrow 1)$  and  $T'(1 \leftrightarrow 0)$ . The distinction in Eq. (8) must be an explicit feature of CVC and should repeat itself for bosons; taking

$$\eta = \frac{1}{2\sqrt{2}} \left[ \begin{array}{c} \bar{n}(1)n(2) + \bar{p}(1)p(2) - \Xi^0(1)\bar{\Xi}^0(2) - \Xi^-(1)\bar{\Xi}^-(2) \\ + (1 \leftrightarrow 2) \end{array} \right] {}^1\chi_0 \quad (9)$$

we find

$$\langle \pi^- | F_- | \eta \rangle = 0 \quad (10)$$

as expected. Any other form for  $\eta$  would have yielded this result, but Eq. (9) is the only one for which  $A\eta = -\eta$ . Although Eq. (10) is hardly accessible to experimental test, detailed measurement of  $\Sigma \rightarrow \Lambda + e + \nu$  will show if  $V_b = 0$ .

For K mesons write

$$\begin{aligned} K^+ &= \frac{1}{2} \left[ \begin{array}{c} d\{\bar{\Lambda}(1)p(2) - \Lambda(1)\bar{\Xi}^-(2)\} + (1 \leftrightarrow 2) \\ + e\{\sqrt{2}\bar{\Sigma}^-(1)n(2) - \bar{\Sigma}^0(1)p(2) - \sqrt{2}\Sigma^+(1)\bar{\Xi}^0(2) \\ - \Sigma^0(1)\bar{\Xi}^-(2)\} \end{array} \right] {}^1\chi_0 \\ K^{0+} &= \frac{1}{2} \left[ \begin{array}{c} d\{\bar{\Lambda}(1)n(2) + \Lambda(1)\bar{\Xi}^0(2)\} + (1 \leftrightarrow 2) \\ + e\{\bar{\Sigma}^0(1)n(2) + \sqrt{2}\bar{\Sigma}^+(1)p(2) - \Sigma^0(1)\bar{\Xi}^0(2) \\ + \sqrt{2}\Sigma^-(1)\bar{\Xi}^-(2)\} \end{array} \right] {}^1\chi_0 \\ K^{0-} &= \overline{K^{0+}}, \quad K^- = -\overline{K^+} \end{aligned} \quad (11)$$

where  $d^2 + 3e^2 = 1$ . Then

$$\begin{aligned}\langle K^- | F_- | K^{0-} \rangle &= \langle K^{0+} | F_- | K^+ \rangle \\ &= V_a [1 - 4de(V_b/V_a) - 4e^2(1 - V_c/V_a)]\end{aligned}\quad (12)$$

leading again to Eq. (8) and CVC, if independence from  $d$  and  $e$  is required.

### 3. OTHER MATRIX ELEMENTS

Now try the Fermi matrix elements for  $\Delta S = \pm 1$  decays. Here the tableau corresponding to Eq. (6) is

$$\begin{aligned}V_d: & (p \rightarrow \Lambda, \bar{\Xi}^- \rightarrow \bar{\Lambda}), -(\bar{\Lambda} \rightarrow \bar{p}, \Lambda \rightarrow \Xi^-) \\ V_e: & (p \rightarrow \Sigma^0, \Sigma^0 \rightarrow \Xi^-), -(\bar{\Sigma}^0 \rightarrow \bar{p}, \bar{\Xi}^- \rightarrow \bar{\Sigma}^0) \\ \sqrt{2}V_e: & (\Sigma^+ \rightarrow \Xi^0, \bar{\Sigma}^- \rightarrow \bar{n}), -(\bar{\Xi}^0 \rightarrow \bar{\Sigma}^+, \bar{n} \rightarrow \bar{\Sigma}^-)\end{aligned}\quad (13)$$

Then

$$\begin{aligned}\langle \pi^- | F_- | K^0 \rangle &= \\ &= \sqrt{2} \langle \pi^0 | F_- | K^+ \rangle = (ad - \sqrt{2}be)V_d + (ae + \sqrt{2}bd + 2\sqrt{2}ce)V_e \\ \sqrt{2} \langle \eta | F_- | K^+ \rangle &= 3eV_e - dV_d.\end{aligned}\quad (14)$$

Any systematic relations based on Eq. (14) must involve the coefficients  $a$  through  $e$  and cannot be expressed exclusively as conditions on  $V_d$  and  $V_e$ . Furthermore, these conditions would be different for  $F_2$  and for  $F_4, \dots$ , so that no universal relation seems feasible.

As an example of a  $G-T$  matrix element, consider  $\rho^+ \rightarrow \pi^0$  for  $N = 1$ . Write

$$\rho^+ = \frac{1}{2} \left[ \begin{aligned} & A\{\bar{n}(1)p(2) - \Xi^0(1)\bar{\Xi}^-(2)\} \\ & + B\{\bar{\Sigma}^-(1)\Lambda(2) - \Sigma^+(1)\bar{\Lambda}(2)\} - (1 \leftrightarrow 2) \\ & + C\{\bar{\Sigma}^-(1)\Sigma^0(2) + \Sigma^+(1)\bar{\Sigma}^0(2)\} \end{aligned} \right] {}^3\chi_1^M \quad (15)$$

where  ${}^3\chi_1^M$  is a symmetric  $\left\{ \begin{smallmatrix} \text{scalar space} \\ \text{vector spin} \end{smallmatrix} \right\}$  function of the baryon and anti-baryon. The non-relativistic  $G-T$  operator in terms of real and isotopic spin operators is

$$\frac{1}{2}[\sigma(1) + \sigma(2)][\tau(1) + \tau(2)] + \frac{1}{2}[\sigma(1) - \sigma(2)][\tau(1) - \tau(2)]. \quad (16)$$

Only the second term in Eq. (16) can flip the spins necessary for



$\rho \rightarrow \pi$ ; because of the opposite symmetries of  $\rho$  and  $\pi$  on  $1 \leftrightarrow 2$ , the associated charge operator is again equivalent to  $T$ . The covariant form under  $E \approx E' \approx m$  (the "basic" mass for all  $0^-$  and  $1^-$  mesons) is thus

$$\left(\frac{M+m}{\sqrt{4EE'}}\right) g' \{ \Phi_\mu^* T_\pm \varphi + \varphi^* T_\pm \Phi_\mu \} l_\mu^\pm. \quad (17)$$

This looks surprisingly like the CVC form but does not represent conserved pseudovector current. The arguments are special to those bosons with  $a_2 \neq 0$ , although all the  $a_{2N}$  terms for such bosons follow Eq. (17); and the factor  $(M+m)$  obtained as normalization to the Dirac wave functions  $\bar{\psi}$  and  $\psi$  is larger than the value  $2m$  appropriate to an interaction between elementary bosons.

According to Eq. (17) the matrix elements for  $\rho^+ \rightarrow \pi^0$ , etc., follow by putting  $(A, B, C)$  for one power of  $a, b, c$  in Eq. (7): viz.,

$$\begin{aligned} \langle \pi^0 | GT | \rho^+ \rangle &= \langle \pi^- | GT | \rho^0 \rangle = \langle \rho^0 | GT | \pi^+ \rangle = \langle \rho^- | GT | \pi^0 \rangle \\ &= \sqrt{2} [AaV'_a - (Bc + bC)V'_b + (bB + cC)V'_c]. \end{aligned} \quad (18)$$

This may be checked directly with Eqs. (5) and (15), remembering to make appropriate corrections to the tableau of Eq. (6) in the case of axial vector interactions  $V'_a$ , etc. There is clearly no possibility for Eq. (18) to be independent of  $A : B : C$  and  $a : b : c$ , which is a necessary condition for current conservation in the present approach.

It would be of interest to compare the  $GT$  matrix element between bosons with  $a_2 = 0$ . An opportunity may arise in  $K \rightarrow 2\pi + e + \nu$ : If the  $2\pi$  are in an  $I = 1$  state, the transition is just  $K \rightarrow \rho$ , given by appropriate substitution in Eq. (18); but the  $I = 0$  state of  $2\pi$  has ideally  $a_2 = 0$ , since the baryon-antibaryon  ${}^3P_0$  state has  $A = -1$ . Without pretending to calculate the matrix element for the second case, we may assume it to be small because of poor overlap of wave functions. Coupled with the  $\Delta I = \frac{1}{2}$  rule, this implies approximate equality for the rates  $K^+ \rightarrow e^+ + \nu + 2\pi$  and  $K^0 \rightarrow e^+ + \nu + 2\pi$ ; and the inhibition of  $2\pi^0$  relative to  $\pi^+\pi^-$  in the first process.

#### 4. ACKNOWLEDGMENT

The author wishes to express his appreciation to the Physics Department at MIT for its hospitality while this note was being written.

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# ON THE LOCALIZATION IN CLASSICAL FIELDS OF ENERGY, MOMENTUM, AND CHARGE \*

GREGOR WENTZEL

*The Enrico Fermi Institute for Nuclear Studies  
and the Department of Physics, The University of Chicago, Chicago, Illinois*

*(Received May 7, 1965)*

## 1. GENERAL CONSIDERATIONS

The object of this study is a classical field, with components  $\psi_\sigma(x_\nu)$  [ $\sigma = 1, 2, \dots n$ ;  $\nu = 1, \dots 4$ ]. The field equations are supposed to be given in terms of a Lagrangian density  $L$  which is a given function of the  $\psi_\sigma$ 's and their first derivatives  $\psi_{\sigma,\nu} = \partial\psi_\sigma/\partial x_\nu$ . For simplicity, we stipulate that  $L$  should not explicitly depend on the coordinates  $x_\nu$ : the fields are source-free. Lorentz-invariant Lagrangian's lead to relativistically covariant field-equations, and a (symmetric) energy-momentum tensor can then be constructed from  $L$  [1]. Among other possible invariance properties of  $L$ , we mention "gauge-invariance of the first kind" which allows to define conserved currents. Such "derived" quantities may have physical significance as potential sources of other fields (gravitational, electromagnetic), but such other fields need not be explicitly invoked for the sake of *defining* the conserved quantities.

One aspect which will concern us here, has been extensively investigated by Belinfante [2]: If one changes the Lagrangian density by adding a divergence

$$L \rightarrow L' = L + \bar{L}, \quad \bar{L} = \sum_\nu \partial A_\nu / \partial x_\nu, \quad (1)$$

the field equations are unaltered, but this may not be so for the derived quantities, even if  $\bar{L}$  has the same invariance properties as  $L$ . The important point is that  $A_\nu$  in (1) may be permitted to depend on both the

\* This work is supported by the U.S. Atomic Energy Commission.

$\psi_\sigma$ 's and their first derivatives  $\psi_{\sigma,\mu}$  without introducing second derivatives in  $\bar{L}$ ; this requires

$$\frac{\partial A_\nu}{\partial \psi_{\sigma,\mu}} + \frac{\partial A_\mu}{\partial \psi_{\sigma,\nu}} = 0. \quad (2)$$

Then, the Lagrangians  $L$  and  $L'$  are still equivalent as far as the free-field equations are concerned, but they usually lead to different expressions for the energy-momentum tensor or the current densities. This ambiguity, as Belinfante has proved, does not affect the integrated quantities (integrals over 3-space), viz. total energy, total linear and angular momentum, and total charge. But their densities, i.e. their localization in 3-space, cannot be uniquely specified in terms of the Lagrangian if suitable  $L$ -transformations of the kind (1), (2) can be set up.

There is, then, a family of Lagrangian densities, none of them preferable with respect to the free-field equations, but suggesting different possible interactions with gravitational and electromagnetic fields. How does "physical reality" make a choice? Or, to ask a more modest question, is it possible to single out a specific  $L'$  by an *objective criterion* which promises to be of general validity? The interactions derived from such a distinguished  $L'$  might then deserve the epithet "*minimal*" (which has been much used recently in a rather haphazard manner).

## 2. SIMPLE EXAMPLES

We first discuss *current densities*, as the simpler objects. The notation  $x_4 = ict$  will be used.

With the use of complex conjugate fields  $\psi_\sigma, \psi_\sigma^*$ ,  $L$  is written as a bilinear expression invariant under the "gauge transformation of the first kind":

$$\psi_\sigma \rightarrow e^{i\alpha} \psi_\sigma, \quad \psi_\sigma^* \rightarrow e^{-i\alpha} \psi_\sigma^* \quad (\alpha = \text{const}). \quad (3)$$

As is well known, a conserved current  $j_\nu$  is then definable (for the limit of vanishing electromagnetic field) by

$$j_\nu = -i \sum_\sigma \left( \psi_\sigma \frac{\partial L}{\partial \psi_{\sigma,\nu}} - \psi_\sigma^* \frac{\partial L}{\partial \psi_{\sigma,\nu}^*} \right), \quad (4)$$

$$\sum_\nu \partial j_\nu / \partial x_\nu = 0 \quad (5)$$



( $j_4/ic$  = charge density). Now, we subject  $L$  to the transformation (1), with (2) assumed valid for  $\psi_\sigma^*$  as well as  $\psi_\sigma$ . Moreover, the gauge transformation (3) should leave  $A_v$  invariant. Then, the change induced in (4) by the  $L$ -transformation is easily found to be

$$j_v \rightarrow j'_v = j_v + \bar{j}_v, \quad \bar{j}_v = \sum_\mu \partial P_{\mu v} / \partial x_\mu, \quad (6)$$

$$P_{\mu v} = -i \sum_\sigma \left( \psi_\sigma \frac{\partial A_\mu}{\partial \psi_{\sigma, v}} - \psi_\sigma^* \frac{\partial A_\mu}{\partial \psi_{\sigma, v}^*} \right) = -P_{v\mu}. \quad (7)$$

The skew-symmetry of  $P_{\mu v}$  follows from (2). The current density  $j_v$  may be attributed to an electric and magnetic polarization.  $\bar{j}_4$  is a 3-space divergence and therefore does not change the total charge.

For the *scalar* field, there exists no 4-vector  $A_v$  obeying (2), and the definition (4) is then unique. But already for the Dirac *spinor* field (spin  $\frac{1}{2}$ ), we may introduce

$$A_v = \lambda \sum_\mu \bar{\psi} [\gamma_v, \gamma_\mu] \frac{\partial \psi}{\partial x_\mu} \quad (\lambda = \text{const.}), \quad (8)$$

$$\bar{L} = \sum_v \frac{\partial A_v}{\partial x_v} = \lambda \sum_{\mu v} \frac{\partial \bar{\psi}}{\partial x_v} [\gamma_v, \gamma_\mu] \frac{\partial \psi}{\partial x_\mu} \quad (9)$$

(in customary notation, writing out the derivatives of the spinors for clarity). Then, according to (7):

$$P_{\mu v} = i\lambda \bar{\psi} [\gamma_v, \gamma_\mu] \psi. \quad (10)$$

Not surprisingly, this is the polarization caused by a "Pauli magnetic moment," with its arbitrary factor  $\lambda$ .

Another instructive example is the (complex) *vector* field ( $\sigma = 1, \dots, 4$ ;  $\psi_4 = i\psi_0$ ,  $\psi_4^* = -\text{c.c. of } \psi_4$ ). We start from the Lagrangian

$$L = -\frac{1}{2} \sum_{\mu v} \left( \frac{\partial \psi_v^*}{\partial x_\mu} - \frac{\partial \psi_\mu^*}{\partial x_v} \right) \left( \frac{\partial \psi_v}{\partial x_\mu} - \frac{\partial \psi_\mu}{\partial x_v} \right) - m^2 \sum_v \psi_v^* \psi_v. \quad (11)$$

A four-vector  $A_v$ , obeying (2) and invariant under (3), is

$$A_v = \frac{1}{2} \lambda \sum_\mu \left( \frac{\partial \psi_\mu^*}{\partial x_\mu} \psi_v - \frac{\partial \psi_v^*}{\partial x_\mu} \psi_\mu \right) + \text{c.c.} \quad (12)$$

Then

$$\bar{L} = \lambda \sum_{\mu\nu} \left( \frac{\partial \psi_\mu^*}{\partial x_\mu} \frac{\partial \psi_\nu}{\partial x_\nu} - \frac{\partial \psi_\nu^*}{\partial x_\mu} \frac{\partial \psi_\mu}{\partial x_\nu} \right). \quad (13)$$

The resulting ambiguity in the current density is well known also in this case [3]. Indeed,

$$P_{\mu\nu} = i\lambda(\psi_\nu^* \psi_\mu - \psi_\mu^* \psi_\nu) \quad (14)$$

is again attributable to an excess magnetic moment of the vector meson.

Regarding Belinfante's *energy-momentum-stress tensor*  $T_{\mu\nu}$ , formulae similar to (6), (7), valid for general fields, can be derived; they involve the matrices which characterize the transformation properties, under infinitesimal Lorentz-transformations, of the  $\psi$ -field. Since only the special case of the *vector* field will be relevant for the following discussion, we merely state the result for this case. Taking again the expression (13) for  $\bar{L}$ , we find for the change in  $T_{\mu\nu}$ , induced by the transformation (1), the following value:

$$\begin{aligned} \bar{T}_{\mu\nu} = \bar{L}\delta_{\mu\nu} + \lambda \mathcal{R} \left\{ \sum_\lambda \left[ \frac{\partial \psi_\lambda^*}{\partial x_\mu} \frac{\partial \psi_\nu}{\partial x_\lambda} + \frac{\partial \psi_\lambda^*}{\partial x_\nu} \frac{\partial \psi_\mu}{\partial x_\lambda} + \right. \right. \\ \left. \left. - 2 \frac{\partial \psi_\mu^*}{\partial x_\lambda} \frac{\partial \psi_\nu}{\partial x_\lambda} + \psi_\lambda^* \frac{\partial}{\partial x_\lambda} \left( \frac{\partial \psi_\mu}{\partial x_\nu} + \frac{\partial \psi_\nu}{\partial x_\mu} \right) \right] - 2m^2 \psi_\mu^* \psi_\nu \right\}. \quad (15) \end{aligned}$$

[ $\mathcal{R}$  means "real part", or more precisely:  $\mathcal{R}A = \frac{1}{2}(A + A^*)$ . Terms vanishing according to the field equations (in particular  $\sum_\nu \partial \psi_\nu / \partial x_\nu = 0$ ) have been omitted.] The tensor (15) is symmetric and obeys the continuity equation  $\sum_\mu \partial \bar{T}_{\mu\nu} / \partial x_\mu = 0$ . One can also verify that the components  $\bar{T}_{4\nu}$  are expressible as 3-space divergences, so as to give no contribution to the total energy and momentum.

### 3. WHAT IS A MINIMAL INTERACTION?

First, we want to point out that simplicity arguments are of dubious value. Such an argument may have some justification in the case of the Dirac electron (Pauli moment = 0), but already in the vector case it becomes a matter of taste. Looking at  $L' = L + \bar{L}$ , with  $L$  and  $\bar{L}$  given by (11) and (13), what  $L'$  is the "simplest"? Certainly,  $\lambda = 0$

is simple, but  $\lambda = 1$  seems equally simple since the second term in (13) just cancels the corresponding term in (11). A general criterion for *all* fields can hardly be established in such a fashion, in the framework of Lagrangian theory.

A more promising suggestion comes from the *energy density* ( $-T_{44}$ ) of the *vector* field (charged or neutral). It is well known that this density as constructed (following Belinfante) from the Lagrangian  $L$  (11) is *manifestly positive-definite*. Eq. (15) shows that *this property is lost* if  $L \rightarrow L + \bar{L}$  with an arbitrary value of  $\lambda$ . We do not here discuss the question what values of  $\lambda$  might be allowable under the requirements of physics. However, the case  $\lambda = 0$  is here clearly distinguished and may serve to define a "minimal" interaction with the gravitational field. It is plausible that the same value  $\lambda = 0$  should be demanded in (14) also; this would give the Lagrangian density a more than mathematical meaning.

This criterion, as it stands, is of course not applicable to the Dirac spinor field, with its states of "positive and negative energy". Two possibilities suggest themselves if one wants a criterion of the same general nature as for the vector field. One might consider the theory *quantized* according to the Pauli exclusion principle, which is well known to make the energy positive-definite (and the charge indefinite). But even within the unquantized ("c-number") version, a particularly simple (though rather mathematical) criterion can be set up. In Dirac's original theory, a density which is manifestly positive-definite is the "charge density"  $\psi^*\psi = \bar{\psi}\gamma_4\psi$ ; this property is however immediately lost if one adds the divergence of a polarization, according to (6) and (10). The absence of a Pauli moment would then naturally characterize the "minimal" interaction with electromagnetic fields (in agreement with common usage).

Are these criteria, based on the signs of  $T_{44}$  and  $\psi^*\psi$ , generalizable to fields representing particles of spin  $> 1$  (integral or half-odd)? This is very doubtful because, in spite of wider possibilities in defining  $T_{\mu\nu}$  and  $j_\nu$  [4], no positive-definite densities have been identified. Although it may be worthwhile further to explore this question, it must be said that this approach can hardly be expected to lead to an entirely satisfactory definition of minimality.

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## BOTTLES FOR NEUTRONS

LESLIE L. FOLDY

*Case Institute of Technology, Cleveland, Ohio*

*(Received May 17, 1965)*

One can conceive of many experiments in nuclear and high energy physics where it would be desirable to have available a gas of free neutrons of appreciable density. Many years ago the author posed to himself the question whether it is at least *theoretically* feasible to construct a "bottle" to hold neutrons. Rather remarkably, within the context in which the question was asked, the answer turned out to be affirmative, although this is not to be taken to mean that the technical feasibility in the future, if not the present, is assured. As will be evident in what follows, many of the principal problems are cryogenic in character – and the author is certainly not sufficiently informed to discuss these in a knowledgeable way – so that it may well turn out that the *technical* solution of these problems can be demonstrated to be impossible on *theoretical* grounds. But in any case there is some interesting physics in the question and this is a pleasant opportunity to commit some of the ideas involved to record.

One must first define what one means by a neutron "bottle". It will be considered here to consist of a cavity in a material substance such that neutrons filling this cavity will, under appropriate circumstances, be unable to escape through the walls at least for a time of the order of the neutron beta-decay lifetime which is 12 minutes. Thus we are indeed considering a bottle in the ordinary sense of the word, and neutrons spatially confined by the action of their own mutual gravitational field (neutron stars) will not be considered to be "bottled" as far as the arguments to follow are concerned.

The first problem is clearly to find a suitable material from which to construct the walls of the bottle. Nature seems to dictate virtually a unique choice for this substance. Any material which captures neutrons will in general lead to a lifetime for radiative capture which is much shorter than the beta-decay lifetime of the neutron unless the

bottle is enormous in size. The only known stable substance which does not capture neutrons is  $\text{He}^4$  so that this must be our choice. To form a cavity surrounded by helium the latter must clearly be in a condensed state, and since no substance can be used inside the cavity to maintain pressure on the helium, one is limited to liquid helium, and superfluid helium at that. Our bottle, and hence the neutrons contained in it, must therefore be at very low temperatures, below the condensation temperature for helium at low pressure which is  $2.2^\circ\text{K}$ . Actually considerably lower temperatures are required as will become evident below.

The next question is clearly whether helium will form a barrier to neutrons or whether neutrons will simply pass through walls composed of it. The pertinent datum here is the potential energy of a neutron inside liquid helium, or, what is equivalent, the index of refraction of liquid helium for cold (long wave-length) neutrons. We ignore the neutron-electron interaction which is negligible for this consideration, in which case this index of refraction is determined by the scattering length of the  $\text{He}^4$  nucleus for slow neutrons. This scattering length is known to be [1]

$$a = 2.4 \times 10^{-13} \text{ cm}, \quad (1)$$

which corresponds to a repulsive potential. In fact, the s-wave scattering of neutrons by  $\text{He}^4$  nuclei up to energies of several Mev is identical with what would be calculated if the nuclei were simply hard spheres of a radius given by (1). Now neutrons whose energy in free space is  $E = \hbar^2 k_0^2 / 2M$ , where  $M$  is the mass of the neutron, will have a wave number  $k$  inside helium substance given by [2]

$$k^2 = k_0^2 + 4\pi na \quad (2)$$

where

$$n = 1.82 \times 10^{22} \text{ atoms/cm}^3 \quad (3)$$

is the number density of helium atoms in liquid helium. Eq. (2) is just the usual expression for the index of refraction of the substance since it relates the wavelength of neutrons of the same energy (frequency) inside and outside the substance. Rewritten it provides the energy-momentum relation for neutrons in helium:

$$E = \hbar^2 k^2 / 2M + V \quad (4)$$

where

$$V = 4\pi n a \hbar^2 / 2M = 1.12 \times 10^{-8} \text{ eV}, \quad (5)$$

is then the potential energy of a neutron in helium. Since  $V$  is positive neutrons with kinetic energy less than  $V$  cannot penetrate into liquid helium from free space, except, of course, for the exponentially damped quantum-mechanical penetration into classically forbidden regions.

In view of (4) a degenerate neutron gas with Fermi energy less than  $V$  at absolute zero will be "contained" by a bottle with liquid helium walls. Even at finite temperatures, provided that they are substantially less than  $10^{-4} \text{ }^\circ\text{K}$ , this containment will be still possible but with some evaporation of neutrons over the barrier potential associated with the tail of the Fermi distribution.

The limitation on the Fermi energy of the neutron gas imposes an upper limit to the density of neutrons which can be contained in the bottle assuming a temperature much lower than  $10^{-4} \text{ }^\circ\text{K}$ . The Fermi energy of the gas is equal to  $V$  at a neutron density of

$$N = (4\pi n a)^3 / 6\pi^2 = 2.2 \times 10^{14} \text{ neutrons/cm}^3. \quad (6)$$

This is quite a respectable density being about  $10^{-5}$  that of ordinary gases at STP, and about  $10^5$  times the neutron density in a high flux nuclear reactor. In fact it is amusing to note that as a consequence of the beta decay of the neutron, such a gas would have a specific activity of  $10^5$  curies/cm<sup>3</sup>.

A further question which requires consideration is the required thickness of the helium walls in order that there is not too high a rate of leakage of neutrons through the helium wall by the process of quantum mechanical tunneling through the potential barrier. We may estimate this for a situation in which the Fermi energy of the neutron gas is one-half that given in Eq. (5) which corresponds to a neutron density about one-third that given by Eq. (6). In this case the neutrons at the Fermi surface have a velocity of about 100 cm/sec and hence, for a bottle whose internal dimensions are of the order of centimeters, these neutrons will make of the order of 100 collisions with the walls per second. To keep the lifetime for leakage through the barrier of the same order as the beta-decay lifetime of the neutron then requires that the probability for a neutron penetrating the barrier on a single en-

counter with the wall be less than  $10^{-5}$ . This will indeed be the case if the wall thickness  $d$  satisfies the condition

$$d \gtrsim (\hbar^2/MV)^{\frac{1}{2}} = 6 \times 10^{-5} \text{ cm.} \quad (7)$$

One should also consider the question of thermal evaporation over the barrier as described earlier, but this can always be kept to a sufficiently low value by reducing the temperature sufficiently; a temperature of the order of  $10^{-5}$  °K would suffice. Of course, the lower the required temperature the more difficult is the practical achievement of an operational neutron bottle but is not necessarily relevant to its theoretical feasibility.

Certainly an important further consideration is whether there is any way of actually enclosing a volume with a liquid helium film of the requisite thickness demanded by (7). One's first thought might be to employ the well-known phenomenon that any surface in contact with bulk liquid helium is covered with a thin film of the liquid [3]. Unfortunately observations on such films indicate that their thickness at a few centimeters above the bulk liquid is only of the order of one-tenth that required by Eq. (7). Other (very uncertain) possibilities might consist in attempting to increase this film thickness through the fountain effect (thermomechanical effect) by maintaining a small positive temperature gradient in the upward direction along the walls of the vessel in which the liquid helium is contained, or by the use of the hydrodynamic phenomenon popularly known as the "teapot effect" [4]. The latter effect, if it occurs in superfluid helium at all, could be exploited by allowing the helium liquid to flow downward through a tube which contains two constrictions, the enlarged portion between them serving as the actual cavity. The teapot effect would then manifest itself in the fact that below the upper constriction the fluid would continue to adhere to the walls and flow down in a film covering the wall of the enlarged portion, instead of separating from the surface in the form of a jet or droplets. Perhaps more ingenious methods may be devised to solve this difficult problem.

It is easy to think of many other problems which would arise in the practical construction of a neutron bottle, not least of which are the cryogenic problems involved in the dissipation of the very considerable heat deposited in the walls of the cavity by the electrons emitted in the



beta-decay of the enclosed neutrons. However, we have achieved some measure of the conditions which must be attained. Briefly summarized we may say that a cavity in liquid helium with helium wall thickness greater than  $6 \times 10^{-5}$  cm and *maintained* at a temperature below  $10^{-5}$  °K would be capable of holding a neutron gas with a density of the order of  $10^{14}$  neutrons/cm<sup>3</sup> with a loss rate which is comparable with that arising from spontaneous neutron decay. Could these conditions be achieved one would indeed have a neutron "bottle". This says nothing about how such a neutron bottle is to be "filled", but this must be left as an exercise to the interested reader; this is not meant to imply that it is a trivial problem.

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# MULTIPOLE RADIATION

KURT GOTTFRIED

*Cornell University, Ithaca, New York*

*(Received May 17, 1965)*

## 1. INTRODUCTION

One-photon states of rather large angular momentum play an important role in many nuclear phenomena. Formulae for transition amplitudes involving such states were first derived by Blatt and Weisskopf [1], and by Wallace [2]. Innumerable applications of these techniques to a wide variety of problems in nuclear spectroscopy have been made in the past fifteen years. One of the earliest and most significant results was the Weisskopf estimate [3] of the one-particle transition amplitude for multipole radiation of arbitrary rank and parity.

The original derivations of the multipole fields and moments are rather lengthy and depend on a number of special devices [4]. In this note we wish to show that a fairly concise and straightforward development is possible if one uses the notion of helicity states introduced by Jacob and Wick [5, 6]. Needless to say, we have no new results to report. This, therefore, is an "Afterlude" and not a "Prelude" in physics! We hope that some readers will find it to be an instructive exercise in quantum mechanical engineering.

In order to make the discussion reasonably self-contained, we begin by reminding the reader of some important properties of rotations, and then derive the basic formula of Jacob and Wick. It will be noted that the actual derivation of the multipole formulae and angular distributions, which begins in Sec. 5, is really quite brief; much of the following is devoted to a resumé of standard results.

## 2. ROTATION MATRICES

Let  $|a\rangle$  be an arbitrary state, and  $|a; R\rangle$  the same state rotated through  $R$ . The rotation  $R$  may be parametrized by three Euler angles  $\alpha$ ,  $\beta$ , and  $\gamma$ . These states are connected by a unitary operator  $U(R)$ :

$$U(R)|a\rangle = |a; R\rangle. \quad (1)$$

In terms of the Euler angles  $U = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$ , where  $\mathbf{J}$  is the total angular momentum of the system [7]. The matrix elements of  $U$  between total angular momentum eigenstates are

$$D_{mm'}^j(R) = \langle jm|U(R)|jm'\rangle. \quad (2)$$

In the sequel we shall only require the  $D$ -matrices with  $\gamma = 0$ . For the sake of brevity we then designate the remaining two angles by a unit vector  $\hat{\mathbf{k}}$ . The orientation of  $\hat{\mathbf{k}}$  is given by the polar angle  $\beta$  and the azimuth  $\alpha$ . Thus we shall write  $U(\alpha\beta 0) \equiv U(\hat{\mathbf{k}})$ , and also  $D_{mm'}^j(\alpha\beta 0) \equiv D_{mm'}^j(\hat{\mathbf{k}})$ .

The  $D$ -matrices satisfy the following important identities [8]:

$$\int d\hat{\mathbf{k}} D_{m\lambda}^j(\hat{\mathbf{k}})^* D_{m'\lambda}^{j'}(\hat{\mathbf{k}}) = \frac{4\pi}{2j+1} \delta_{jj'} \delta_{mm'}, \quad (3)$$

$$\int d\hat{\mathbf{k}} D_{m\lambda}^j(\hat{\mathbf{k}})^* D_{m'\lambda}^{j'}(\hat{\mathbf{k}}) Y_{LM}^*(\hat{\mathbf{k}}) = \frac{\sqrt{4\pi(2L+1)}}{2j+1} \langle j'm'LM|jm\rangle \langle j'\lambda L0|j\lambda\rangle. \quad (4)$$

Here  $\int d\hat{\mathbf{k}}$  indicates an integration over the unit sphere, and we use the Condon-Shortley conventions for the Clebsch-Gordan coefficients and spherical harmonics.

### 3. ONE-PHOTON HELICITY STATES

We wish to build one-photon states having definite total angular momentum quantum numbers  $j$  and  $m$ . Following Jacob and Wick, we shall construct these from one-photon states  $|k\hat{\mathbf{z}}; \lambda\rangle$  of linear momentum  $k\hat{\mathbf{z}}$  and helicity  $\lambda$  ( $\lambda = \pm 1$ ), where  $\hat{\mathbf{z}}$  is a unit vector along the  $z$ -direction. This state is already an eigenstate of  $J_z$  with eigenvalue  $\lambda$ . Because of this the decomposition of  $|k\hat{\mathbf{z}}; \lambda\rangle$  into angular momentum eigenstates  $|k; jm\rangle$  only contains terms with  $m = \lambda$ :

$$|k\hat{\mathbf{z}}; \lambda\rangle = \sum_{j=1}^{\infty} |k; j\lambda\rangle \langle k; j\lambda|k\hat{\mathbf{z}}; \lambda\rangle. \quad (5)$$

A one-photon state of helicity  $\lambda$  propagating in the direction  $\hat{\mathbf{k}}$  can be obtained from  $|k\hat{\mathbf{z}}; \lambda\rangle$  by the rotation  $U(\hat{\mathbf{k}})$ , i.e.,  $|\mathbf{k}; \lambda\rangle = U(\hat{\mathbf{k}})|k\hat{\mathbf{z}}; \lambda\rangle$ . (Recall that the helicity is a pseudoscalar.) When we apply this rotation to (5) and use (2) we obtain

$$|\mathbf{k}; \lambda\rangle = \sum_{jm} |k; jm\rangle D_{m\lambda}^j(\hat{\mathbf{k}}) \langle k; j\lambda|k\hat{\mathbf{z}}; \lambda\rangle.$$

We may now extract the sought-after angular momentum eigenstate with the help of the orthogonality relation (3):

$$|k; jm\lambda\rangle = [(2j+1)/4\pi]^{\frac{1}{2}} \int d\hat{k} D_{m\lambda}^j(\hat{k})^* |k; \lambda\rangle. \quad (6)$$

This state obviously has helicity  $\lambda$ , and we have therefore inserted this quantum number into the ket symbol. In writing (6) we have adjusted  $\langle k; j\lambda|k\hat{z}; \lambda\rangle$  to conform with the normalization conventions

$$\langle k; \lambda|k'; \lambda'\rangle = \delta_{\lambda\lambda'} \delta(k-k'), \quad (7)$$

$$\langle k; jm\lambda|k'; j'm'\lambda'\rangle = \frac{\delta(k-k')}{kk'} \delta_{jj'} \delta_{mm'} \delta_{\lambda\lambda'}. \quad (8)$$

The amplitude for finding a photon of linear momentum  $k$  in a state having specified total angular momentum quantum numbers is therefore

$$\langle k; \lambda|k'; jm\lambda'\rangle = \frac{\delta(k-k')}{kk'} \delta_{\lambda\lambda'} \sqrt{\frac{2j+1}{4\pi}} D_{m\lambda}^j(\hat{k})^*. \quad (9)$$

#### 4. THE VECTOR POTENTIAL

We can now construct an expression for the vector potential in terms of operators that destroy and create the helicity states  $|k; jm\lambda\rangle$ . We begin once more with the linear momentum representation. Here the vector potential assumes the form

$$A(r) = (8\pi^3)^{-\frac{1}{2}} \int \frac{d^3k}{\sqrt{2k}} [e^{-ik \cdot r} e_{k\lambda}^* a_{\lambda}^\dagger(k) + \text{h.c.}], \quad (10)$$

where  $a_{\lambda}^\dagger(k)$  creates the state  $|k; \lambda\rangle$  when acting on the vacuum, and  $e_{k\lambda}$  is a circular polarization unit vector.

We can define operators  $a_{jm\lambda}^\dagger(k)$  that create the states  $|k; jm\lambda\rangle$  when acting on the vacuum by means of the linear transformation (6). When these are inserted into (10) we obtain the desired expression for the vector potential:

$$A(r) = \sum_{jm, \lambda=\pm 1} \frac{(j+\frac{1}{2})^{\frac{1}{2}}}{4\pi^2} \int_0^\infty \frac{k^2 dk}{\sqrt{2k}} [f_{jm}^\lambda(k, r) a_{jm\lambda}^\dagger(k) + \text{h.c.}], \quad (11)$$



where

$$f_{jm}^\lambda(k, \mathbf{r}) = \int d\mathbf{k} \mathbf{e}_{\mathbf{k}\lambda}^* e^{-i\mathbf{k} \cdot \mathbf{r}} D_{m\lambda}^j(\hat{\mathbf{k}}). \quad (12)$$

The vector fields defined in this last equation are closely related to the vector spherical harmonics used by Blatt and Weisskopf [1]. The transformation that takes us from (10) to (11) also provides us with the expansion of a plane electromagnetic wave in terms of spherical waves (see the alternative discussion in ref. 1, p. 807, and the more closely related treatment of Rose [7], p. 137).

## 5. EMISSION AMPLITUDES

Let us compute the amplitude for the process where a photon of momentum  $\mathbf{k}$  and helicity  $\lambda$  is emitted while the source undergoes the transition  $|a\rangle \rightarrow |b\rangle$ . This amplitude is

$$A_{ba}(k\lambda) = -\langle b; \mathbf{k}; \lambda | \int \mathbf{A}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}) d^3r | a; 0 \rangle, \quad (13)$$

where  $\mathbf{j}(\mathbf{r})$  is the current density in the source system, and  $|a; 0\rangle$  is the product of the source state  $|a\rangle$  and the electromagnetic vacuum state. Upon inserting the complete set of one-photon angular momentum eigenstates into this matrix element, we find,

$$\begin{aligned} A_{ba}(k\lambda) &= \\ &= - \sum_{jm} \int_0^\infty k'^2 dk' \langle b; \mathbf{k}; \lambda | k'; jm\lambda \rangle \langle b; k; jm\lambda | \int \mathbf{A}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}) d^3r | a; 0 \rangle. \end{aligned}$$

The transformation function that appears here is given by (9). With the help of (11) one easily reduces the remaining matrix element to an element referring only to the source operators and states. One thereby obtains

$$A_{ba}(k\lambda) = - \frac{1}{16\pi^2 \sqrt{\pi k}} \sum_{jm} (2j+1) D_{m\lambda}^j(\hat{\mathbf{k}})^* \langle b | T_{j, -m}^\lambda | a \rangle, \quad (14)$$

where

$$T_{jm}^\lambda = \int \mathbf{j}(\mathbf{r}) \cdot \mathbf{f}_{j, -m}^\lambda(k, \mathbf{r}) d^3r. \quad (15)$$

Eq. (14) gives the angular distribution of photons of helicity  $\lambda$  emitted

in the transition  $|a\rangle \rightarrow |b\rangle$ . The angular functions that appear in (14) are related to the spherical harmonics by

$$D_{m, \pm 1}^j(\hat{\mathbf{k}})^* = \sqrt{\frac{4\pi}{j(j+1)(2j+1)}} \left( \frac{-m}{\sin \beta} \mp \frac{\partial}{\partial \beta} \right) Y_{jm}(\hat{\mathbf{k}}).$$

Because the separate terms in (14) correspond to the emission of photons into angular momentum states with the quantum numbers  $(j, m)$ , the quantity  $T_{jm}^\lambda$  must be the  $m^{\text{th}}$  component of a tensor operator of rank  $j$ . When the states of the source are angular momentum eigenstates, the sum in (14) is restricted by the selection rules  $|J_a - J_b| \leq j \leq J_a + J_b$ , and  $M_b = M_a - m$ .

## 6. MULTIPOLE MOMENTS

The tensor operators  $T_{jm}^\lambda$  are closely related to the conventional electromagnetic multipole moments. The precise relationship becomes apparent if one determines the behavior of  $T_{jm}^\lambda$  under reflections. Let  $P$  be the unitary reflection operator. Under a reflection the current density  $\mathbf{j}(\mathbf{r})$  transforms as follows:

$$P\mathbf{j}(\mathbf{r})P^{-1} = -\mathbf{j}(-\mathbf{r}).$$

When we reflect (15) we therefore obtain

$$PT_{jm}^\lambda P^{-1} = -\int \mathbf{j}(\mathbf{r}) \cdot \mathbf{f}_{j, -m}^\lambda(k, -\mathbf{r}) d^3r,$$

where

$$\mathbf{f}_{j, -m}^\lambda(k, -\mathbf{r}) = \int d\hat{\mathbf{k}} e_{-\mathbf{k}\lambda}^* e^{-i\mathbf{k} \cdot \mathbf{r}} D_{-m\lambda}^j(-\hat{\mathbf{k}}). \quad (16)$$

But  $D_{mm'}^j(-\hat{\mathbf{k}}) = D_{mm'}^j(\pi + \alpha, \pi - \beta, 0) = (-1)^j D_{m, -m'}^j(\hat{\mathbf{k}})$ . The relationship between  $e_{-\mathbf{k}\lambda}$  and  $e_{\mathbf{k}\lambda}$  is obtained as follows. We first define  $e_{\mathbf{k}\lambda}$  for all  $\mathbf{k}$  by means of

$$e_{\mathbf{k}\lambda} = \sum_{\mathbf{v}} e_{\mathbf{v}} D_{\mathbf{v}\lambda}^1(\hat{\mathbf{k}}), \quad (17)$$

where  $e_0 = \hat{\mathbf{z}}$ ,  $e_{\pm 1} = \mp(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})/\sqrt{2}$ . As a consequence of this definition  $e_{-\mathbf{k}\lambda} = -e_{\mathbf{k}, -\lambda}$ . Returning to (16) we conclude that

$$\mathbf{f}_{jm}^\lambda(k, -\mathbf{r}) = -(-1)^j \mathbf{f}_{jm}^{-\lambda}(k, \mathbf{r}). \quad (18)$$

The reflection property of the operators  $T_{jm}^\lambda$  is therefore

$$PT_{jm}^\lambda P^{-1} = (-1)^j T_{jm}^{-\lambda}. \quad (19)$$

The fact that  $T_{jm}^\lambda$  is transformed into  $T_{jm}^{-\lambda}$  is simply a consequence of the helicity being a pseudoscalar.

When one deals with transitions between states of definite parity, it is convenient to have operators that transform into themselves under reflection. With this end in view we define two new operators by

$$T_{jm}^\pm = T_{jm}^1 \pm T_{jm}^{-1}. \quad (20)$$

The reflection character of these operators is given by

$$PT_{jm}^\pm P^{-1} = \pm (-1)^j T_{jm}^\pm. \quad (21)$$

By definition, the parity changes by  $(-1)^j$  in an electric multipole transition of order  $2^j$ , and by  $(-1)^{j+1}$  in a magnetic transition of this order. It is therefore clear that  $T_{jm}^+$  and  $T_{jm}^-$  are proportional to the electric and magnetic multipole moments, respectively.

By combining (12), (15), and (18), we can write an explicit formula for the electric and magnetic multipole moments, viz.

$$T_{jm}^\pm = \int d\hat{\mathbf{k}} d^3r (j \cdot \mathbf{e}_{\mathbf{k}1}^*) D_{-m1}^j(\hat{\mathbf{k}}) [e^{-i\mathbf{k} \cdot \mathbf{r}} \mp (-1)^j e^{i\mathbf{k} \cdot \mathbf{r}}]. \quad (22)$$

If one expands the plane waves into spherical waves, i.e.,

$$e^{-i\mathbf{k} \cdot \mathbf{r}} = 4\pi \sum_{LM} i^{-L} j_L(kr) Y_{LM}(\hat{\mathbf{r}}) Y_{LM}^*(\hat{\mathbf{k}}),$$

and recalls (17), one can carry out the integration over  $\hat{\mathbf{k}}$  by using (4). The triangular inequalities implicit in the Clebsch-Gordan coefficients then require that  $L = j-1, j, j+1$ . Because of the last factor in the integrand of (22), the harmonics of rank  $L = j \pm 1$  contribute to the electric multipoles, and the remaining term with  $L = j$  contributes to the magnetic multipoles.

When the wavelength  $1/k$  is long compared to the dimension of the source one can approximate the spherical Bessel function by

$$j_L(kr) \simeq \frac{(kr)^L}{(2L+1)!!}. \quad (23)$$

In this situation the electric multipole moment is dominated by the term with  $L = j-1$ . The Weisskopf estimates follow more or less directly from these remarks.

Compact expressions for the multipole moments can be obtained when the long wavelength approximation (23) is valid [1, 2]. We first carry out the  $\hat{k}$ -integration in the manner already indicated, and find

$$T_{jm}^+ = 2(4\pi)^{\frac{3}{2}} i^{2m-j+1} \frac{k^{j-1}}{(2j-1)!!} \sqrt{\frac{j+1}{2(2j-1)(2j+1)}} \times \\ \times \sum_{Mv} \langle jm1v | j-1M \rangle \int \mathbf{j} \cdot \mathbf{e}_v^* r^{j-1} Y_{j-1M}(\hat{\mathbf{r}}) d^3r. \quad (24)$$

By using standard results concerning the spherical harmonics and a table of Clebsch-Gordan coefficients one can show that

$$\nabla(r^j Y_{jm}) = -(2j+1) \sqrt{j/2j-1} \sum_{Mv} \mathbf{e}_v^* \langle jm1v | j-1M \rangle r^{j-1} Y_{j-1M}.$$

This is precisely the sum that appears in (24). After integrating by parts and using the continuity equation we therefore obtain

$$T_{jm}^+ = -2(4\pi)^{\frac{3}{2}} \frac{k^{j-1}}{(2j+1)!!} i^{2m-j+1} \sqrt{\frac{j+1}{2j(2j+1)}} \times \\ \times \int r^j Y_{jm}(\hat{\mathbf{r}}) \dot{\rho}(\mathbf{r}) d^3r, \quad (25)$$

where  $\dot{\rho}$  is the time derivative of the charge density. Aside from a factor of  $ik$ , the integral in (25) is the electric multipole operator  $Q_{jm}$  of Blatt and Weisskopf. The evaluation of the magnetic multipole moment requires the use of the identity

$$LY_{jm} = \sqrt{j(j+1)} \sum_{Mv} \mathbf{e}_v^* \langle jm1v | jM \rangle Y_{jM}.$$

The long wavelength approximation to the magnetic multipole moment is then

$$T_{jm}^- = 2(4\pi)^{\frac{3}{2}} \frac{k^j}{(2j+1)!!} i^{2m-j+1} \sqrt{\frac{1}{2j(2j+1)(j+1)}} \times \\ \times \int r^j Y_{jm}(\hat{\mathbf{r}}) \nabla \cdot (\mathbf{r} \times \mathbf{j}) d^3r. \quad (26)$$



The integral in this expression equals  $-(j+1)M_{jm}$ , where  $M_{jm}$  is the magnetic multipole moment of Blatt and Weisskopf.

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# INELASTIC SCATTERING AND ASSOCIATED GAMMA RADIATION

DAVID R. INGLIS

*Argonne National Laboratory, Argonne, Illinois*

*(Received May 17, 1965)*

When alphas at a few tens of MeV are scattered by medium-weight nuclei, various phenomena suggest that the scattering nucleus may be treated as approximately black. The very striking elastic-inelastic phase rule of Blair is an example [1]. It states that the phases of angular scattering patterns are so related that the maxima of the elastic scattering pattern coincide in angle with the minima of the pattern for inelastic scattering with no change of parity in the nuclear excitation. It may be most simply explained on the basis of Fraunhofer diffraction from a black disk for elastic scattering and from a ring aperture at the edge of the black disk for inelastic scattering. The ring aperture is a model for the requirement that the scattered particle must pass close to the edge of the nucleus to excite it by means of short-range forces. The black disk produces the well-known Fraunhofer pattern, the variation of the scattered amplitude with angle having an almost sinusoidal nature (actually a Bessel function). This can be obtained from the Huygens-Kirchhoff integration over the plane outside the disk, and the integration over the ring may be obtained by differentiation of this with respect to the radius [2]. The derivative operator puts the inelastic amplitude out of phase with the elastic one.

It would be dangerous to conclude from this that the nucleus is black. Machine computations with the DWBA typically find several sets of parameters for the real well depth and the absorption parameter (imaginary well depth) that agree with the scattering experiments. Nevertheless, the black-nucleus approximation seems to represent the essential feature.

It is of interest to examine the phenomenon of inelastic scattering more closely, taking into account the mechanism of the nuclear excitation but still retaining the simplification of an approximately black nucleus.

A particularly interesting and simple indication of the nature of the nuclear excitation is found in the inelastic alpha scattering yielding the lowest rotational states of the simple deformed nuclei  $C^{12}$ ,  $Mg^{24}$ , and  $Si^{28}$ . These  $4n$  nuclei lack the complicating feature of low intrinsic excitations. Here the inelastic scattering and subsequent gamma emission involves a  $0-2-0$  transition and the angular pattern of the  $E2$  gammas has the symmetry of a four-petal rosette or a four-bladed

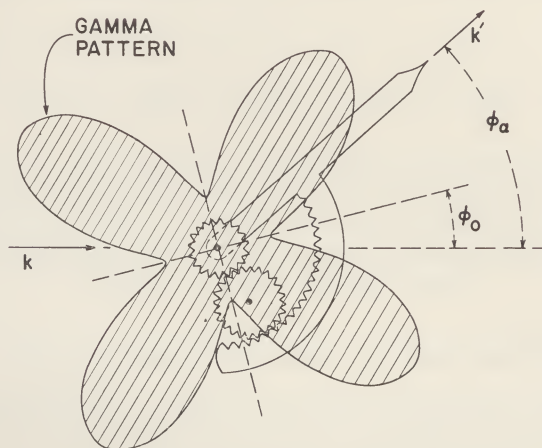


Fig. 1. The lever representing the direction  $\phi_\alpha$  of the alpha counter pivots on the same axis as does the central small gear on which the fan is mounted. The shape of the fan (cut out of transparent plastic) represents the intensity of the coincident gammas as a function of angle in the reaction plane. The alpha lever carries a large-radius gear section. The off-center small gear, with its axle also fixed to the base plate, then acts as a step-up reversing gear and, as the alpha direction is slowly rotated, the gamma-orientation angle  $\phi_0$  rotates rapidly in the reverse sense, the ratio of angular speeds being the ratio of the radii of the larger and smaller concentric gears.

fan. The orientation of the pattern shows the remarkable reverse rotation illustrated crudely by the mechanical gadget sketched in Fig. 1. The effect is observed only for alpha angles between about  $20^\circ$  and  $90^\circ$  and the "gear ratio" is not constant (Fig. 2), but the rapid reverse rotation is the most striking aspect of the phenomenon. It is so striking as to call for a very simple answer to the question: What is there in quantum mechanics to replace the reversing gear?

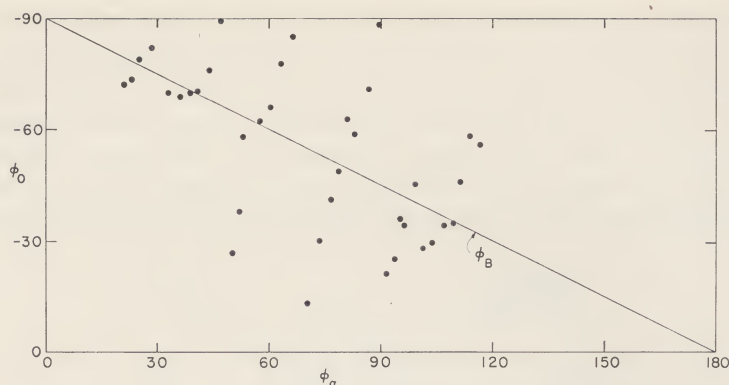


Fig. 2. Observed orientation of the gamma-ray pattern for  $\text{Mg}^{24}$ , according to Ref. 4. The scale for  $\phi_0$  is negative, so the positive slope of the lines of points indicates a reverse rotation. The diagonal line is the recoil direction (for  $k \approx k'$ ) and represents the experimental trend discussed earlier (Ref. 7) because high intensity makes the points there least difficult to observe.

#### AN INADEQUATE THEORY: THE FUZZY-PROFILE MODEL

For the sake of perspective, let us first examine an oversimplified treatment of inelastic scattering – one that is not adequate to account for the gamma rotation. The Fraunhofer theory of elastic scattering is a profile theory: the absorbing nucleus is represented by its silhouette, a black disk with a sharp cut-off at the edge. As a slight relaxation of this idealization, consider instead a fuzzy profile, the edge of which is a “grey wedge” fading from white to black. For simplicity of drawing and of conveying the main ideas, we shall consider scattering in only two dimensions rather than three. The nucleus is then a circle of radius  $a$  and its profile is a line segment along the  $y$  axis from  $-a$  to  $a$ . The transmission through the “grey wedge” at the edge may be described by the function

$$\mathcal{J}(y) \begin{cases} = 1 & \text{for } |y| > a \\ = e^{-g(a-|y|)} & \text{for } |y| < a \end{cases} \quad (1)$$

With this as the weighing function, a Fraunhofer-Kirchhoff integration along the  $y$  axis gives an explicit elastic scattering amplitude  $a_{el}$ . For  $g \gg 1/a$  and  $g \gg K = k \sin \theta = k'_y$ , one finds as the leading



terms in the elastically scattered intensity

$$a_{\text{el}} a_{\text{el}}^* \propto \frac{\sin^2 Ka}{K^2} + \frac{\cos^2 Ka}{g^2}. \quad (2)$$

A similar integration is encountered when inelastic scattering is treated by the distorted-wave Born approximation (DWBA) to the Schrödinger equation. The line-segment obstacle representing the nucleus has a small but finite thickness,  $X$ . Within it at a given value of  $y$  just less than  $a$ , the incident wave  $\xi_i$  is attenuated towards the right and the final wave  $\xi_f$  is attenuated towards the left. The product of the amplitudes of the two waves is thus nearly constant inside the obstacle and roughly equal to  $\mathcal{J}(y)$ , the amplitude of  $\xi_i$  at the right-hand edge, where  $\xi_f$  has unit amplitude. Thus the same "grey wedge"

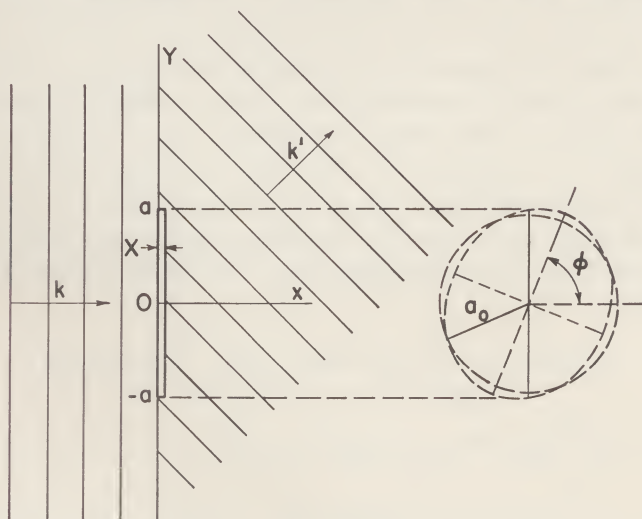


Fig. 3. Dependence of the profile on the collective nuclear coordinate  $\phi$ .

at the edge of the nucleus applies to elastic scattering, where it adds only a small second term in Eq. (2), and to the DWBA, where it provides a finite amplitude of the wave functions within the nucleus.

The excitation of a rotational state of a deformed nucleus may be represented in the profile model by letting the dimension  $a$  of the profile be a function of the collective nuclear rotation coordinate  $\phi$  as indicat-

ed in Fig. 3. The small elliptical deformation of the nucleus from circular shape is given by

$$r(\xi) = a_0 + a_1 \cos 2(\psi - \phi), \quad (3)$$

where the angle  $\phi$  locates the major axis. The width of the profile is then taken to be the dimension,  $2a(\phi)$ , of the ellipse along the  $y$  axis, with

$$a(\phi) = a_0 + a_1 \cos 2(\frac{1}{2}\pi - \phi) = a_0 - a_1 \cos 2\phi. \quad (4)$$

In the DWBA, the distorted waves are solutions of the wave equation for the undeformed nucleus, in our case the thin rectangle of width  $2a_0$  and thickness  $X \rightarrow 0$ . The deformation then provides the perturbation term of the Hamiltonian, the short-range ( $\delta$ -function) interaction  $\mathcal{H}'$  between the scattered particle and the nuclear matter (or lack thereof) representing deformation [the term in  $a_1$  of Eq. (4)]. The matrix elements for the inelastic transition contains this multiplied by the initial and final nuclear wave functions and the product of the incoming and outgoing waves. We confine attention to the simplest case of small deformation  $a_1 \gg g^{-1}$  and thus  $\mathcal{J}(y) \approx 1$  in the region through which  $a(\phi)$  varies. The rotational nuclear wave functions are  $e^{im\phi}$ , and this is 1 for the ground state  $m = 0$ . A typical matrix element is then

$$\begin{aligned} \langle m | \mathcal{H}' | 0 \rangle &= \int_0^{2\pi} d\phi e^{-im\phi} \Sigma_{\pm} \left[ \pm \int_{\pm a_0}^{\pm a(\phi)} dy \mathcal{J}(y) e^{iKy} \right] \\ &= - \int_0^{2\pi} d\phi e^{-im\phi} a_1 \cos 2\phi [e^{iKa} + e^{-iKa}] \\ &= -a_1 \cos Ka \Sigma_{\pm} \int_0^{2\pi} d\phi e^{i(m \pm 2)\phi} = -2\pi \delta(m, \pm 2) a_1 \cos Ka. \end{aligned} \quad (5)$$

Thus the intensity of inelastic scattering is proportional to  $\cos^2 Ka$ , and hence is out of phase in  $\theta$  with the elastic scattering [the large term of Eq. (2)] in keeping with the Blair phase rule.

The two degenerate nuclear states  $m = \pm 2$  are excited with the same phase in  $\phi$  and the complete excited nuclear state is

$$u_2(\phi) \approx a_1 \cos Ka (e^{2i\phi} + e^{-2i\phi}) = 2a_1 \cos Ka \cos 2\phi.$$

The probability distribution of the major axis is then

$$u_2 u_2^* \approx a_1^2 \cos^2 Ka \cos^2 2\phi. \quad (6)$$

This has the shape of a four-bladed fan with the blades along the  $x$  and  $y$  axes. The radiation pattern on de-excitation of such a distribution has the same shape but with the blades midway between these, the pattern being rotated by  $\pi/4$ . (The radiation in the direction  $\kappa$  contains a gradient operator normal to  $\kappa$  which, for a bulge of charge moving around a circle, may be expressed as  $(1/R) \cos(\phi - \phi_\kappa) \partial/\partial\phi$ . The derivative operator places the radiation pattern just out of phase with the probability pattern, as in the familiar case of dipole radiation normal to the dipole.)

Thus the radiation pattern is stuck between the axes and does not rotate. If we modify the treatment slightly by placing the line of the profile in the direction of the recoil,  $\mathbf{K} = \mathbf{k}' - \mathbf{k}$ , (which approximately bisects the angle between  $\mathbf{k}$  and  $\mathbf{k}'$  if  $k \approx k'$ , yielding the familiar relation  $K = 2k \sin \frac{1}{2}\theta$ ) the rotation pattern is instead fixed to the slowly-varying recoil direction.

Another treatment that fails to impart a rotation to the gamma pattern for a similar reason is the plane-wave Born approximation (PWBA). The nucleus in two dimensions is treated as a circle plus a slight deformation – not as its profile – and  $\xi_i$  and  $\xi_f$  are plane waves sweeping across the circle. The product of the two plane waves is a plane wave in the recoil direction and this direction is the only preferred axis introduced by the scattering process. The result is symmetric with respect to reflection in that line, and the gamma pattern again is tied to it. Thus the two dimensions of the nucleus and some distortion of the waves appears to be needed to impart the rotation.

#### THE “BEATS” AT THE EDGE OF THE NUCLEUS

The incident distorted wave  $\xi_i$ , which is a distortion of  $e^{i\mathbf{k} \cdot \mathbf{r}}$ , does not stop at the geometric shadow but instead bends around the nuclear surface in the shadow – though with reduced intensity. We assume that its wavelength  $\lambda$  remains unchanged along the surface (though there is probably a fairly uniform reduction of wavelength around the lateral edge). The distorted outgoing wave  $\xi_f$  has a similar shape and has  $k' < k$  and  $\lambda' > \lambda$  because of the inelastic scattering. The product

of the two waves around the lateral edges of the nuclear surface is important in determining the relative phase of the excitation matrix elements. There is always one arbitrary over-all phase having no physical meaning, so we may make the product of the two waves  $e^{ik \cdot r}$  and  $e^{ik' \cdot r}$  real at the center.

If one takes two combs, for example, the spacing of the teeth being slightly greater in one than the other, and looks at a light background through both of them placed together, one may see alternate bands of

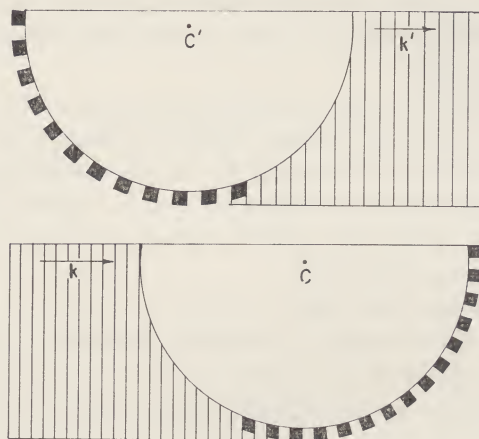


Fig. 4. Do-it-yourself demonstration kit for the reverse rotation of the beats. Instructions: Imagine the upper panel drawn on transparent paper, the center  $C'$  superposed on  $C$ , and a pin stuck through the two centers. Then rotating  $k'$  from  $\phi_\alpha = 45^\circ$  to  $90^\circ$  causes reverse rotation of the beats between the gear teeth. The ratio of the tooth spacing of the upper panel to that of the lower is  $4/3$ .

light and dark, known as "beats". If the comb with the greater spacing ("wavelength") is moved slowly to the right, the beats move rapidly to the left. Similarly, if  $k$  is horizontal to the right and  $k'$  upward to the right at an angle  $\phi_\alpha$  as in Fig. 1, we consider at the lower edge of the nucleus the product of the two waves  $\xi_i$  and  $\xi_f^*$ . A given phase of the product moves rapidly to the left as the waves of  $\xi_f$  move to the right with increase of  $\phi_\alpha$  (Fig. 4). This, together with a similar situation at the upper edge, is the signal that is carried to the nucleus to make



possible the rapid reverse rotation of the gamma pattern, as has been pointed out in an earlier letter [8].

Two lines through the center, one normal to  $k$  and the other to  $k'$ , delimit a "shadow sector" at the bottom of the nucleus (region B in Fig. 5) and a "bright sector", that is "seen" by both waves, at the

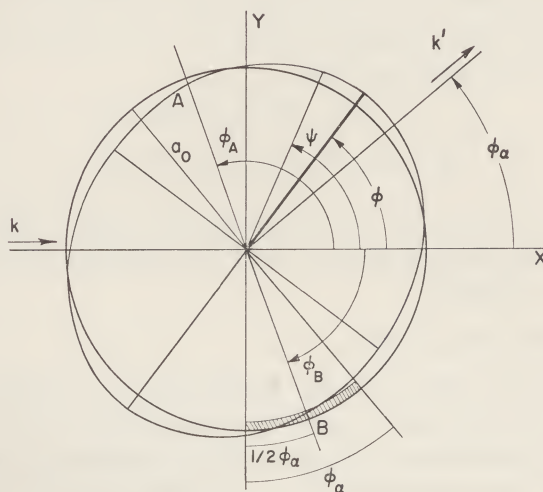


Fig. 5. Angles used in the two-dimensional treatment.

top (region A). On these two lines the respective waves have the same phase as at the center (their product being unity), so the phase of the product of the two waves at an angle  $\psi$  in region B, for example, is determined by the distances along the edge from those lines, i.e.,

$$P_B = [\xi_i(\psi)\xi_f(\psi)]_B \propto e^{i[(k+k')a_0(\frac{1}{2}\phi_\alpha) + (k-k')a_0(\psi - \phi_B)]}, \quad (7)$$

with

$$\phi_B = -\frac{1}{2}\pi + \frac{1}{2}\phi_\alpha.$$

The reverse "beat" phenomenon may be seen clearly in this factor. In the first term in the exponent – the term giving the phase at the mid-point of the shadow sector – the alpha angle  $\phi_\alpha$  enters with a large positive coefficient  $(k+k')$ . The second term gives the variation of phase throughout region B and in it  $\psi$  has a small positive coefficient  $(k-k')$ . Thus a large decrease in  $\psi$  is required to compensate a small

increase in  $\phi_\alpha$ . In region A, we have the same phase factor as in region B but with  $a_0 \rightarrow -a_0$  and  $-\frac{1}{2}\pi \rightarrow \frac{1}{2}\pi$ . As for their amplitudes, we expect that the products of the waves reach a maximum near the midpoints of the bright and dark sectors and assume that this variation may reasonably be represented by a Gaussian factor in each case. Aside from this, we assume that the product tends to be weaker in region B than in A, and hence introduce a factor  $W < 1$  in region B. The products in the two regions are thus

$$\begin{aligned} P_A &= e^{-\frac{1}{2}iS\phi_\alpha} e^{-iD(\psi-\phi_A)} e^{-\beta(\psi-\phi_A)^2}, \\ P_B &= W e^{\frac{1}{2}iS\phi_\alpha} e^{iD(\psi-\phi_B)} e^{-\beta(\psi-\phi_B)^2}, \end{aligned} \quad (8)$$

with  $S = (k+k')a_0$ ,  $D = (k-k')a_0$ , and  $\phi_A = \phi_B + \pi$ . The signs of  $S$  and  $D$  change between regions A and B because the direction of the radius  $a_0$  is reversed.

#### THE NUCLEAR EXCITATION

The perturbation term  $\mathcal{H}'$  in the Hamiltonian is a  $\delta$ -function interaction between the alpha at  $\mathbf{r}$  and the nuclear matter in the "bulge" described by the term in  $a_1$  of Eq. (3). The deformation  $a_1$  is small and the strength of the interaction at  $\phi$  is proportional to the magnitude of that term, i.e., to the thickness of the bulge at  $\phi$ . The contribution of the interaction in the region B to the matrix element exciting the rotational state  $e^{im\phi}$  is thus

$$\begin{aligned} \langle m | \mathcal{H}' | 0 \rangle_B &= a_1 \int d\phi e^{-im\phi} \int d\psi \cos 2(\psi - \phi) P_B(\psi - \phi_B) \\ &= \frac{1}{2} a_1 W e^{\frac{1}{2}iS\phi_\alpha} \int d\psi' e^{-\beta\psi'^2 + iD\psi'} \int_0^{2\pi} d\phi \left[ \sum_{n=\pm 2} e^{in(\phi-\psi)} \right] e^{-im\phi} \\ &= \pi \delta(m, \pm 2) a_1 W e^{\frac{1}{2}iS\phi_\alpha} e^{-im\phi_B} \int d\psi' e^{-\beta\psi'^2 + (D-m)\psi'} \\ &= \delta(m, \pm 2) C_1 w_m e^{i(\frac{1}{2}S\phi_\alpha - m\phi_B)}, \end{aligned} \quad (9)$$

where

$$\psi' = \psi - \phi_B, \quad C_1 = \pi a_1 \sqrt{(\pi/\beta)}, \quad w_m = W e^{-(m-D)^2/4\beta}. \quad (10)$$

In the corresponding contribution from region A,  $w_m$  is replaced by  $v_m$  which lacks the factor  $W$  and has the reversed sign of  $D$ :

$$v_m = e^{-(m+D)^2/4\beta}. \quad (11)$$

It is to be noted that  $v_{-2}$  and  $w_2$  are larger than  $v_2$  and  $w_{-2}$ , respectively, because the former contain the larger exponential factor. The inequality

$$e^{-(2-D)^2/4\beta} > e^{-(2+D)^2/4\beta} \quad (12)$$

expresses a tendency for conservation of momentum in the excitation process (but not a very strong tendency because the region of integration is not much longer than  $\lambda$ ). For example,  $v_{-2}$  is large because the state  $m = -2$  has forward momentum in region A to account for part of the momentum lost by the scattered alpha. The ratio of the terms (12) is not greatly different from unity because the excitation energy is much less than the alpha energy,  $D$  being only  $\frac{1}{3}$  to  $\frac{1}{2}$  in typical cases. Of the four coefficients, the smallest is  $w_{-2}$  because it contains both the small factor  $W$  and the smaller exponential factor, so for simplicity we take  $w_{-2} = 0$ . The excited-state wave function, made up of the two degenerate rotational states, is then

$$\begin{aligned} u_{\text{exc}} &= \sum_{m=\pm 2} \langle m | \mathcal{H}' | 0 \rangle e^{im\phi} = C_1 \sum_{m=\pm 2} (v_m e^{-\frac{1}{2}i\gamma} + w_m e^{\frac{1}{2}i\gamma}) e^{im\phi'} \\ &= C_1 e^{-\frac{1}{2}i\gamma} [(v_2 + w_2 e^{i\gamma}) e^{2i\phi'} + v_{-2} e^{-2i\phi'}] \\ &= C_1 e^{-i(\gamma/2 + 2\phi'_0)} [C e^{2i(\phi' - \phi'_0)} + v_{-2} e^{-2i(\phi' - \phi'_0)}], \end{aligned} \quad (13)$$

with  $\gamma = S\phi_\alpha$ ,  $\phi' = \phi - \phi_B$ , and  $\phi'_0 = \phi_0 - \phi_B$ . Here we have emphasized the important relative phase of the two terms by setting

$$v_2 + w_2 e^{i\gamma} = C e^{-4i\phi'_0} \quad (14)$$

where

$$C = (v_2^2 + 2v_2 w_2 \cos \gamma + w_2^2)^{\frac{1}{2}}.$$

The probability distribution of the major axis is then given by

$$\begin{aligned} u_{\text{exc}} u_{\text{exc}}^* / C_1^2 &= C^2 + v_{-2}^2 + C v_{-2} \{e^{4i(\phi - \phi_0)} + \text{c.c.}\} \\ &= \sigma_\alpha + C v_{-2} \{[\cos 2(\phi - \phi_0) + i \sin 2(\phi - \phi_0)]^2 + \text{c.c.}\} \\ &= \sigma_\alpha + 2C v_{-2} [1 - 2 \sin^2 2(\phi - \phi_0)] \\ &= A - B' \sin^2 2(\phi - \phi_0) \quad (B' \text{ positive}), \end{aligned} \quad (15)$$

where c.c. means "complex conjugate". Here the cross section for inelastic scattering of alphas, obtained by integrating over all nuclear

orientations  $\phi$ , is

$$\sigma_\alpha = C^2 + v_-^2 = v_2^2 + w_2^2 + v_-^2 + 2v_2 w_2 \cos \gamma. \quad (16)$$

As already mentioned, the gamma-emission pattern is just out of phase with  $u_{\text{exc}} u_{\text{exc}}^*$ , with maxima and minima interchanged by changing the sign of the term in  $B$ :

$$\sigma_\gamma(\phi_\gamma) \propto A + B \sin^2 2(\phi_\gamma - \phi_0) \quad (B \text{ positive, } B > B'),$$

and this corresponds to the  $\phi_0$  used to express the experimental results.

The striking reverse rotation we are after comes out of the determination of  $\phi_0$  from the real and imaginary parts of Eq. (14):

$$\frac{\sin 4\phi'_0}{\cos 4\phi'_0} = - \frac{w_2 \sin \gamma}{v_2 + w_2 \cos \gamma},$$

$$\phi_0 = \phi_B - \frac{1}{4} \tan^{-1} \frac{\sin \gamma}{v_2/w_2 + \cos \gamma}. \quad (17)$$

The behavior of  $\phi_0$  then depends on the magnitude of the ratio  $v_2/w_2$ . If  $v_2/w_2 > 1$ , the tangent is never infinite and the arctan varies periodically within narrow limits. The approximate conservation of momentum, however, tends to make  $v_2$  small and  $w_2$  large. If  $v_2/w_2 < 1$ , the tangent passes through infinity twice in each cycle of  $\gamma$ . The reciprocal of the denominator alone goes through infinity with the opposite change of sign on successive passages (+ to -, then - to +); but between successive passages the numerator changes sign so the successive passages have the same change of sign and there is a secular increase of the arctangent (Fig. 6). Thus the arctangent increases by  $2\pi$ , making  $\phi_0$  decrease by  $\frac{1}{2}\pi$ , for each increment  $2\pi$  in  $\gamma$ . In typical cases  $\frac{1}{2}S$  is of the order of 10 so  $\gamma \approx 10\phi_\alpha$  and the reverse rotation is rapid, as observed.

Particularly in the excitation of  $M = 2$ , we see that there is competition between the momentum-conserving integral over the shadow region and the "brute force" integral from the bright region and only the momentum-conserving integral is sensitive to phases so as to communicate to the gamma radiation the reverse rotation of the "beats."

Qualitatively, then, here is the quantum-mechanical reversing gear. While the profile model, which neglects the dimension of the nucleus



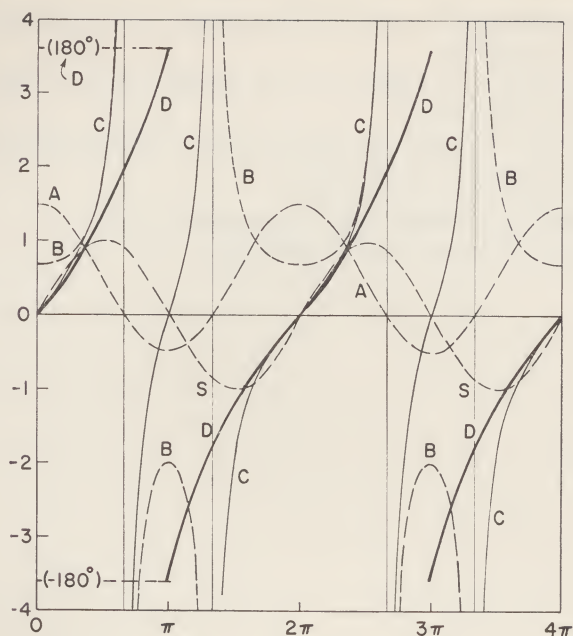


Fig. 6. Determination of the reverse rotation by the factors of Eq. (17) for  $v_2/w_2 = \frac{1}{2}$ . The curve labels are as follows:  $S = \sin \gamma$ .  $A$  is the denominator,  $A = \frac{1}{2} + \cos \gamma$ .  $B$  is its reciprocal and passes through infinitely successively in opposite directions,  $B = 1/A$ .  $C$  is obtained by multiplying  $B$  by  $S$ , and passes through infinity always in the same direction.  $D = \tan^{-1} C$  and continually increases. It is plotted modulo  $2\pi$ , so as to keep it between  $-\pi$  and  $\pi$ , corresponding to the way the experimental points are plotted in Fig. 2.

roughly parallel to the beam direction, accounts for the Blair elastic-inelastic phase rule, we see that momentum transfer in this dimension must be considered in order to understand the nuclear excitation process.

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# ON SYMMETRY TRANSFORMATIONS

G. C. WICK

*Columbia University and Brookhaven National Laboratory*

*(Received May 17, 1965)*

In the theory of symmetry transformations in Quantum Mechanics, a central role is played by "Wigner's theorem" which states that every transformation of the "rays" of a Hilbert space, which preserves the inner product of rays, can be regarded as the result of either a unitary or an anti-unitary transformation of the vectors of the space. This theorem is important, because, although one often says that Quantum Mechanics describes physical states by means of vectors  $\psi$ , a more correct statement is that it assigns to each state a definite "ray"; in other words the vector assigned to a state is defined only up to an arbitrary multiplicative constant  $\lambda$  [1]. Thus, from the assumption that it should be possible to translate the mathematical description of a state, given by an observer, into a description meaningful to another observer using a different reference system, one can only infer the existence of a correspondence between "rays" in Hilbert space. It is therefore important to know that such a correspondence can always be described in terms of a mapping of vectors into vectors, and that this mapping is furthermore linear or antilinear (more concisely: semilinear).

Wigner's theorem, which provides the necessary link, is proved in his well known book [2]. The proof is somewhat involved although each step is really quite elementary; furthermore, a really complete proof is even more involved than indicated in the book [3]. Various learned papers have been published, containing real or alleged improvements of the proof, or giving stronger theorems [4].

The aim of the following considerations is not so much to give a new proof of the theorem as to make it more plausible by relating it to certain facts which are very familiar to physicists. In addition, this will provide an opportunity for some scattered remarks on ray-space.

## 1. INNER PRODUCT OF RAYS

We shall indicate unit vectors by kets, such as  $|\alpha\rangle$ ,  $|\zeta\rangle$ , etc. In particular, in an  $n$ -dimensional vector space, the kets

$$|1\rangle, |2\rangle, \dots, |n\rangle \quad (1)$$

will designate an orthonormal set. The corresponding rays will be indicated by the corresponding letter:  $\alpha$ ,  $\zeta$ , etc. or number 1, 2,  $\dots$ ,  $n$ . Of course the vector  $|\alpha\rangle$  specifies the ray  $\alpha$  completely while  $\alpha$  specifies the vector  $|\alpha\rangle$  only up to a phase factor.

The inner product  $(\alpha\beta)$  of two rays is by definition the absolute value of the scalar product  $\langle\alpha|\beta\rangle$  of the corresponding unit vectors. We also define a "distance"  $\rho_{\alpha\beta}$  between the two rays by the formula

$$\cos(\frac{1}{2}\rho_{\alpha\beta}) = (\alpha\beta) = |\langle\alpha|\beta\rangle| \quad (2)$$

$$0 \leq \rho_{\alpha\beta} \leq \pi.$$

One easily sees that zero distance ( $\rho_{\alpha\beta} = 0$ ) implies that the rays are identical ( $|\alpha\rangle = \lambda|\beta\rangle$ ), while maximum distance ( $\rho_{\alpha\beta} = \pi$ ) means that the two rays are orthogonal. The physical interpretation of the square of the expression (2) as a *transition probability* is too well known to require special comments, but we notice that it is because of this interpretation, that the hypothesis of Wigner's theorem restricts attention to ray-transformations "which preserve the inner product of rays." In the following, this qualification will be tacitly understood whenever a ray-transformation is mentioned.

## 2. TWO-DIMENSIONAL VECTOR SPACE

The main idea of the following proof is, that for the rays of a two-dimensional vector space "Wigner's theorem" is a direct consequence of the elementary geometrical proposition that every transformation of the surface of a sphere, which preserves the arc-distance between points, is either a rotation about the center of the sphere or a pseudorotation. The latter is, of course, a rotation accompanied by a reflection in a plane through the center.

The connection between this elementary fact and Wigner's theorem rests on the following familiar notions. A unit vector in two dimensions



may be written:

$$|\zeta\rangle = e^{iz} \begin{pmatrix} \cos \frac{1}{2}\theta \\ (\sin \frac{1}{2}\theta)e^{i\phi} \end{pmatrix}; \quad 0 \leq \theta \leq \pi \quad (3)$$

In discussing the corresponding ray  $\zeta$ , the phase factor in front may be ignored. The ray may, therefore, be represented by a point on the surface of a sphere, with polar coordinates  $\theta$ ,  $\phi$  or, alternatively, cartesian coordinates:

$$n_i = \langle \zeta | \sigma_i | \zeta \rangle \quad (i = 1, 2, 3) \quad (4)$$

where the  $\sigma_i$ 's are the usual Pauli matrices. The correspondence is one-to-one.

The expressions (3) and (4) are quite familiar to physicists; if (3) is a "spinor" the unit vector  $\mathbf{n}$  gives the "direction of the spin." They are also familiar in classical optics, where they are used to describe the polarization states of a photon by a point on the "Poincaré sphere."

As is well known, a unitary transformation of  $|\zeta\rangle$  corresponds to a rotation of  $\mathbf{n}$ ; on the other hand, changing  $|\zeta\rangle$  to its complex conjugate corresponds to the transformation:

$$n_1 \rightarrow n_1, \quad n_2 \rightarrow -n_2, \quad n_3 \rightarrow n_3 \quad (6)$$

i.e. a reflection in the "13" plane. More generally an anti-unitary transformation of  $|\zeta\rangle$  (a unitary transformation accompanied by complex conjugation) corresponds to a pseudo-rotation of the vector  $\mathbf{n}$ . As is well known, *the converse of these statements is also true.*

An elementary calculation, starting from (3), shows that the arc-distance between the representative points of two rays, or in other words the angle between the corresponding two spin-directions, is equal to the ray-distance  $\rho$ , as defined in Eq. (2).

For a two-dimensional vector-space, therefore, Wigner's theorem is the exact equivalent of the elementary geometrical proposition mentioned earlier.

It is now easy to base the proof for the general case on the result just obtained. For convenience, let us state this result as follows. Let  $T$  be a

distance-preserving transformation of the rays of a two-dimensional vector space. Let  $|1\rangle$  and  $|2\rangle$  be orthonormal vectors,  $|\zeta\rangle$  a general unit vector of the space:

$$|\zeta\rangle = \lambda_1|1\rangle + \lambda_2|2\rangle. \quad (5)$$

The theorem states that, for a suitable choice of the phases [5] of the unit vectors  $|1'\rangle$ ,  $|2'\rangle$  representing the transformed rays  $1' = T1$ ,  $2' = T2$ , the ray  $\zeta' = T\zeta$  is represented (for all values of  $\lambda_1$ ,  $\lambda_2$ ) either by the unit vector

$$(\text{linear case}) \quad |\zeta'\rangle = \lambda_1|1'\rangle + \lambda_2|2'\rangle \quad (6)$$

or by

$$(\text{antilinear case}) \quad |\zeta'\rangle = \lambda_1^*|1'\rangle + \lambda_2^*|2'\rangle. \quad (6a)$$

It is easy to convince ourselves that these formulae will also describe a transformation of the rays of a two-dimensional space into rays of *another* two-dimensional space.

### 3. THE GENERAL CASE

It is now easy to extend the proof to vector spaces of higher dimensionality. We begin by noticing that the idea of *linear dependence* applies to rays as well as to vectors. Obviously the statement that  $m$  unit vectors  $|\alpha\rangle$ ,  $|\beta\rangle$ , ...,  $|\zeta\rangle$  are linearly dependent, remains true after each vector is multiplied by an arbitrary phase factor. It is therefore meaningful to say that the *rays*  $\alpha$ ,  $\beta$ , ...,  $\zeta$  are linearly dependent.

The rays corresponding to vectors of a two-dimensional subspace spanned by two vectors  $|\alpha\rangle$  and  $|\beta\rangle$  will be said to form a linear subspace  $R(\alpha, \beta)$  of ray-space [6]. Any two distinct rays of the subspace will define the same subspace.

If, in particular, we choose  $\alpha$  and  $\beta$  to be orthogonal to each other:

$$(\alpha\beta) = 0 \quad (7)$$

then any ray  $\gamma$  of the subspace satisfies the condition

$$(\alpha\gamma)^2 + (\beta\gamma)^2 = 1 \quad (8)$$

and conversely conditions (7) and (8) are sufficient conditions [7] for linear dependence of  $\gamma$  on  $\alpha$  and  $\beta$ . The corresponding rays  $\alpha'$ ,  $\beta'$ ,  $\gamma'$

in a ray-transformation will obviously satisfy the same conditions. They will therefore be linearly dependent. This argument is easily extended to a linear combination of  $m$  orthonormal vectors, so that in conclusion *our ray transformations map linear subspaces into subspaces of the same dimensionality*.

Let now a ray transformation  $T$  be given in a vector space  $\mathcal{H}$ . In the following:  $|\zeta\rangle \rightarrow |\zeta'\rangle$  indicates that  $|\zeta\rangle$  and  $|\zeta'\rangle$  are representative unit vectors of corresponding rays:  $\zeta' = T\zeta$ , without implying that their phases are chosen in some particular way. We need not even assume that  $|\zeta\rangle$  and  $|\zeta'\rangle$  are unit vectors, but we shall always assume they have the same norm. Thus, if  $|\zeta\rangle$  belongs to  $R(\alpha, \beta)$ :  $|\zeta\rangle = \lambda|\alpha\rangle + \mu|\beta\rangle$ , and

$$|\alpha\rangle \rightarrow |\alpha'\rangle, \quad |\beta\rangle \rightarrow |\beta'\rangle$$

we can write

$$\lambda|\alpha\rangle + \mu|\beta\rangle \rightarrow \lambda'|\alpha'\rangle + \mu'|\beta'\rangle. \quad (9)$$

From the conservation of inner products [8] one has:

$$|\lambda| = |\lambda'|; \quad |\mu| = |\mu'|. \quad (9a)$$

Notice that this is a weaker statement than Eq. (6) or (6a), but on the other hand it does not require a particular choice of phase for  $|\alpha'\rangle$  and  $|\beta'\rangle$ , and it does not distinguish between linear and antilinear case. In Eq. (9), the phase of the coefficient  $\lambda'$  may be chosen arbitrarily, that of  $\mu'$  is then determined by the ray transformation.

We construct the semilinear vector transformation in  $\mathcal{H}$ , corresponding to  $T$  as follows. We select, as one usually does, some unit vector  $|1\rangle$  and determine once and for all in some arbitrary way the phase of the corresponding vector  $|1'\rangle$ . Then if  $|k\rangle$  is any vector orthogonal to  $|1\rangle$ , and  $|k\rangle \rightarrow |k'\rangle$ , the phase of  $|k'\rangle$  may be fixed by the observation that  $T$  transforms the subspace  $R(1, k)$  into  $R(1', k')$ ; the result at the end of Section 2 indicates that the phase of  $|k'\rangle$  can be chosen in such a way that for  $R(1, k) \rightarrow R(1', k')$  the transformation is described by a formula analogous to either (6) or (6a). This leaves the possibility open that the transformation be linear for some value of  $|k\rangle$ , antilinear for some other value. We shall see that this cannot happen.

Consider a combination of three orthonormal vectors

$$|\zeta\rangle = \lambda_1|1\rangle + \lambda_2|2\rangle + \lambda_3|3\rangle. \quad (10)$$

If one of the three constants  $\lambda_i$  is zero,  $\zeta$  belongs to one of the subspaces  $R(1, 2)$ ,  $R(1, 3)$ ,  $R(2, 3)$ . The phases of  $|2\rangle$  and  $|3\rangle$  in:

$$|2\rangle \rightarrow |2'\rangle; \quad |3\rangle \rightarrow |3'\rangle$$

have already been fixed as described above. Let us assume, for example, that the transformation has the *linear* form, Eq. (6) for  $R(1, 2)$  and likewise for  $R(1, 3)$ . There is a remarkably simple connection between the transformation of the general ray  $\zeta$ , Eq. (10) and the transformations of the subspaces  $R(i, j)$ . The following calculation is based on a joint application of (6) and of the weaker form (9) (9a). In the first line for example we express  $\zeta$  as a ray of  $R(\alpha, 3)$  where  $\alpha$  is a ray of  $R(1, 2)$ . We use (6) in  $R(1, 2)$  and (9) (9a) in  $R(\alpha, 3)$ . Thus

$$\begin{aligned} |\zeta\rangle &= (\lambda_1|1\rangle + \lambda_2|2\rangle) + \lambda_3|3\rangle \\ &\rightarrow (\lambda_1|1'\rangle + \lambda_2|2'\rangle) + \lambda'_3|3'\rangle; |\lambda'_3| = |\lambda_3|. \end{aligned} \quad (11a)$$

Similarly:

$$\begin{aligned} |\zeta\rangle &= (\lambda_1|1\rangle + \lambda_3|3\rangle) + \lambda_2|2\rangle \\ &\rightarrow (\lambda_1|1'\rangle + \lambda_3|3'\rangle) + \lambda'_2|2'\rangle; |\lambda'_2| = |\lambda_2|. \end{aligned} \quad (11b)$$

The two expressions on the right-hand side must represent the same ray, hence  $\lambda_1 : \lambda_1 = \lambda_2 : \lambda'_2 = \lambda_3 : \lambda_3$  or (for  $\lambda_1 \neq 0$ )

$$|\zeta\rangle \rightarrow |\zeta'\rangle = \lambda_1|1'\rangle + \lambda_2|2'\rangle + \lambda_3|3'\rangle \quad (11c)$$

a *linear law for the general vector*  $|\zeta\rangle$ .

The remaining calculations are obvious modifications of the preceding one and will be left to the reader. First Eq. (11c) may be related, by writing  $|\zeta\rangle = \lambda_2|1\rangle + (\lambda_1|2\rangle + \lambda_3|3\rangle)$  etc. to the transformation of  $R(2, 3)$ . The result, as expected, is that (11c) holds also for  $\lambda_1 = 0$ .

In a similar way, if we assume an antilinear law in  $R(1, 2)$  and  $R(1, 3)$  we find an antilinear law in the whole subspace. But the assumption that  $R(1, 2)$  transforms, say, linearly and  $R(1, 3)$  antilinearly, is incompatible with any transformation law of  $R(2, 3)$ .

We see, therefore, that there is a simple connection between the transformation laws in all the subspaces  $R(1, k)$  considered before; for every  $R(1, k)$  will be a subspace of some three-dimensional space, Eq. (10), where 2 may be kept fixed while  $k$  is varied. It follows that



$R(1, k)$  transforms in the same way as  $R(1, 2)$ . It may be easily seen that our construction defines uniquely a transformation

$$|\zeta'\rangle = U|\zeta\rangle$$

of the vector space  $\mathcal{H}$ . For any two vectors  $|\zeta\rangle$  and  $|\eta\rangle$ , we may choose  $|2\rangle$  and  $|3\rangle$  in such a way that  $|\zeta\rangle$  and  $|\eta\rangle$  belong to the three-dimensional subspace considered there. It then follows from Eq. (11c), or the analogous equation for the antilinear case, that  $U$  is semilinear, for example, from Eq. (11c)

$$U(\lambda|\zeta\rangle + \mu|\eta\rangle) = \lambda U|\zeta\rangle + \mu U|\eta\rangle. \quad (12)$$

This, then, completes the proof of the theorem.

#### 4. SOME REMARKS ON RAY-SPACE

As we have seen, the metric defined by Eq. (2) is such that a two-dimensional ray-space can be mapped faithfully on the "Poincaré sphere." In this mapping, two orthogonal rays  $\alpha$  and  $\beta$  are mapped on opposite poles on the sphere. It is then obvious why any ray  $\gamma$  which is linearly dependent on  $\alpha$  and  $\beta$  (and which can be represented, therefore, by a point on the sphere) must satisfy Eq. (8) or the equivalent sum-rule for the distances mentioned in [7]. See Fig. 1.

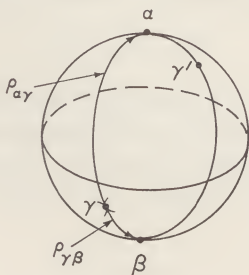


Fig. 1. When  $(\alpha\beta) = 0$ ,  $\alpha$  and  $\beta$  are represented by opposite poles on the sphere. Any point  $\gamma$  on the sphere satisfies Eq. (13) with  $\rho_{\alpha\beta} = \pi$ .

If  $\alpha$  and  $\beta$  are not orthogonal ( $\rho_{\alpha\beta} < \pi$ ) a condition for  $\gamma$ , stronger than linear dependence, is that  $\gamma$  should lie on the geodesic arc connecting  $\alpha$  to  $\beta$ . In this case, of course, the distances satisfy the condition:

$$\rho_{\alpha\beta} = \rho_{\alpha\gamma} + \rho_{\gamma\beta}. \quad (13)$$

See Fig. 2. Conversely one can show, by means of the Gram determinant of the three vectors  $|\alpha\rangle$ ,  $|\beta\rangle$ , and  $|\gamma\rangle$  that condition (13) implies linear dependence [9]. Since then  $\gamma$  belongs to  $R(\alpha, \beta)$  the three rays can be represented by points on a sphere, and Eq. (13) implies that the situation is as described by Fig. 2.

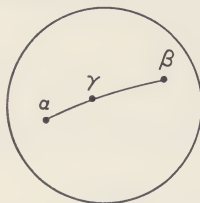


Fig. 2. When  $(\alpha\beta) \neq 0$ , only the points  $\gamma$  on the arc of great circle from  $\alpha$  to  $\beta$  satisfy Eq. (13).

These considerations obviously lead to the notion of geodesic arc in a multidimensional ray-space. If  $\alpha$  and  $\beta$  in Eq. (13) are kept fixed, and  $\gamma$  is allowed to vary, the equation forces  $\gamma$  to follow a one-dimensional path contained in  $R(\alpha, \beta)$ . This path is a "shortest path," in the sense that for any  $\gamma$  *not* on the path one has the inequality:

$$\rho_{\alpha\beta} < \rho_{\alpha\gamma} + \rho_{\gamma\beta}. \quad (14)$$

The notion of geodesic arc could also be introduced from a differential point of view, by noting that Eq. (2) defines in particular a Riemannian metric  $ds^2$  for infinitesimally close rays. Calculation shows that the finite distance, Eq. (2), is simply the integral

$$\rho_{\alpha\beta} = \int_{\alpha}^{\beta} ds \quad (15)$$

calculated along the shortest path from  $\alpha$  to  $\beta$ .

When the two rays  $\alpha$  and  $\beta$  are not orthogonal to each other, there is, as one often says, a certain "coherence" between the physical states represented by  $\alpha$  and  $\beta$ , in the sense that the relative phase of  $|\alpha\rangle$  and  $|\beta\rangle$  may be fixed in a natural way by requiring the scalar product  $\langle\alpha|\beta\rangle$  to be *real* and *positive*. It is easy to verify that, when this is done,

the rays  $\gamma$  of the geodesic arc from  $\alpha$  to  $\beta$  correspond to vectors of the form

$$|\gamma\rangle = \lambda|\alpha\rangle + \mu|\beta\rangle \quad (16)$$

with  $\lambda/\mu$  real and positive. This shows that the geometric notion of geodesic arc in ray-space is connected with the physical notion of coherence of quantum-states.

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- 5) More precisely: the phase of one vector, say  $|1'\rangle$ , can be chosen arbitrarily. Only the *relative* phase of  $|1'\rangle$  and  $|2'\rangle$  matters.
- 6) This subspace is two-dimensional in the sense that two real parameters are needed to specify a ray.
- 7) Compare Bargmann, loc. cit., § 3. Notice that in terms of distances between rays, condition (7) and (8) read:

$$\rho_{\alpha\beta} = \pi; \quad \rho_{\alpha\gamma} + \rho_{\gamma\beta} = \pi.$$

The meaning of these equations is apparent in the geometrical picture, see Sect. 4.

- 8) One can easily see that this is true for any value of  $(\alpha\beta) \neq 1$ . We shall, however, only use these equations for the case where  $|\alpha\rangle$ ,  $|\beta\rangle$  (and consequently  $|\alpha'\rangle$ ,  $|\beta'\rangle$ ) form an orthonormal system. In this case (9a) is obvious.
- 9) We shall not go into this, but shall notice simply that the Gram determinant is zero in the case of linear dependence, otherwise it is positive. It is easy to see then when (13) is satisfied, the Gram determinant can only be zero.

# SHADOW SCATTERING BY ATOMS

H. A. BETHE

*Cornell University, Ithaca, New York*

*(Received May 20, 1965)*

Mott and Massey [1] have pointed out that the scattering of electrons of medium energy (100–1000 eV) by atoms shows a strong maximum at small angles [2]. The scattering at larger angles can be calculated in the Born approximation for light atoms like He, and by calculating phase shifts in a static potential for heavier atoms [3]. This corresponds essentially to the Hartree approximation; the atom is supposed to be unaffected by the incident electron. To explain the strong forward maximum, however, Mott and Massey invoke the “polarization” of the atom by the incident electron [4]; when this is included, good quantitative agreement with experiment is obtained for He.

The calculation by Massey and Mohr [4] leads to an effective “polarization potential” acting on the incident electron,

$$v_P = -\frac{ie^2 a_0}{2kr^3} + \mathcal{O}\left(\frac{1}{k^2 r^4}\right) \quad (1)$$

where  $a_0$  is the Bohr radius. The term of order  $r^{-4}$  is real. The leading term, being purely imaginary, was interpreted by Mott and Massey as follows: It “corresponds to an absorption potential. It may, perhaps, be interpreted as due to the loss of electrons from the incident beam by inelastic scattering”.

We want to show here that this interpretation is not just “perhaps” but is correct, and leads to an easy understanding as well as a simple calculation of the forward maximum. This maximum, then, is the atomic counterpart of the well-known “shadow scattering” in nuclear physics.

In the presence of inelastic scattering, the elastically scattered amplitude is given by [5]

$$f = \frac{i}{2k} \sum_l (2l+1)(1-\eta_l)P_l(\cos \theta) \quad (2)$$



where  $\eta_l$  is the amplitude of the outgoing wave with angular momentum  $l$ . We may write this

$$\eta_l = a_l e^{2i\delta_l}. \quad (3)$$

In the absence of inelastic scattering, the real factor  $a_l$  is equal to one; in the presence of such scattering,  $a_l < 1$ . The contribution of the partial wave  $l$  to the inelastic cross section is [5]

$$\sigma_{\text{inel}, l} = \pi \lambda^2 (2l+1)(1-a_l^2). \quad (4)$$

Now it is well known that electrons may cause excitation and ionization of atoms even if they pass at large impact parameters  $b = \lambda l$ , namely at  $b \gg a$  where  $a$  is the atomic radius. The reason for this is, of course, the very long range of the Coulomb interaction between external and atomic electron. Therefore  $\sigma_{\text{inel}, l}$  will remain appreciable for very large  $l$ , for which  $\delta_l$  is negligible. Therefore, for

$$l = bk \gg a \quad (5)$$

we have essentially

$$\eta_l = a_l. \quad (6)$$

These are typical conditions for shadow scattering. Moreover, since  $a_l < 1$  up to very large  $l$ , this shadow scattering is concentrated at very small angles.

A good estimate for  $\sigma_{\text{inel}, l}$  may be obtained by Williams' classical method [6]. Since the impact parameter is large, the incident electron may be treated classically, and its trajectory as a straight line. Its interaction with the atomic electrons is

$$V = \sum_i \frac{e^2}{|\mathbf{r}_0 - \mathbf{r}_i|} \approx e^2 \frac{\mathbf{r}_0}{r_0^3} \cdot \sum_i \mathbf{r}_i \quad (7)$$

where  $\mathbf{r}_0$  is the position of the incident electron and  $i$  labels the atomic electrons. We write

$$\mathbf{r}_0 = \mathbf{b} + \mathbf{v}t \quad (8)$$

where  $\mathbf{b}$  is the impact parameter,  $\mathbf{b} \cdot \mathbf{v} = 0$ . Only the component  $\mathbf{b}$  in the numerator  $\mathbf{r}_0$  of (7) contributes appreciably to the transition amplitude [7]; choosing  $\mathbf{b}$  in the  $X$ -direction, (7) becomes

$$V = \frac{e^2 b}{(b^2 + v^2 t^2)^{3/2}} X; \quad X = \sum_i x_i. \quad (9)$$

The transition amplitude from the initial atomic state 0 to the final state  $n$  is, apart from a factor of modulus 1,

$$\langle n|T|0\rangle = \hbar^{-1} \int_{-\infty}^{\infty} \langle n|V|0\rangle e^{i\omega t} dt \quad (10)$$

where

$$\hbar\omega = E_n - E_0. \quad (11)$$

Integration over  $t$  gives in sufficient approximation

$$\begin{aligned} \langle n|T|0\rangle &= \frac{2e^2}{\hbar v b} \langle n|X|0\rangle & \text{if } b < v/\omega \\ &= 0 & \text{if } b > v/\omega. \end{aligned} \quad (12)$$

The probability of *all* inelastic transitions is then

$$1 - a_i^2 = \sum_n |\langle n|T|0\rangle|^2 = \left(\frac{2e^2}{\hbar v b}\right)^2 \sum_n \langle n|X|0\rangle^2. \quad (13)$$

For a given  $b$ , the condition (12),  $\omega < v/b$ , permits only excited states up to a certain energy. However, the matrix element  $\langle n|X|0\rangle$  is very small for high excitation, therefore the sum in (13) may be extended over all states  $n$  and evaluated by closure,

$$\begin{aligned} \sum_n \langle n|X|0\rangle^2 &= \langle 0|X^2|0\rangle = \frac{1}{3} \langle 0|\mathbf{R}^2|0\rangle \\ &= \frac{1}{3} \langle 0| \sum_i \mathbf{r}_i^2 + \sum_i \sum_{j \neq i} \mathbf{r}_i \cdot \mathbf{r}_j |0\rangle \end{aligned} \quad (14)$$

assuming that the ground state is isotropic. Neglecting the correlation  $\mathbf{r}_i \cdot \mathbf{r}_j$ , and assuming that the state of the atom can be described by electron orbitals, we have

$$\sum = \frac{1}{3} \sum_{\alpha} z_{\alpha} r_{\alpha}^2. \quad (15)$$

Here  $z_{\alpha}$  is the number of electrons in shell  $\alpha$ , and  $r_{\alpha}^2$  is the mean square radius of that shell. The atomic radius may be taken to be equal to the largest  $r_{\alpha}$  which we shall call  $r_1$ , thus:

$$a = r_1 \equiv (r_{\alpha})_{\max}. \quad (15a)$$

The expression on the right hand side of (15) is proportional to the diamagnetism of the atom. In any case, it is clear that for the important values of  $b$  ( $\geq a$ ), (13) will be very small so that we may write

$$1 - a_l \approx \frac{1}{2}(1 - a_l^2) = \frac{2}{3} \left( \frac{e^2}{\hbar v b} \right)^2 \sum_{\alpha} z_{\alpha} r_{\alpha}^2 \quad (16)$$

or, in terms of  $l = kb$ :

$$1 - a_l = \frac{2}{3} \sum_{\alpha} \frac{z_{\alpha} r_{\alpha}^2}{a_0^2} \frac{1}{l^2} \equiv \frac{C}{l^2} \quad (17)$$

where  $a_0 = \hbar^2/me^2$  is the Bohr radius. This is a very simple result; it is remarkable that it is independent of  $v$ . It is valid, for the contribution from shell  $\alpha$ , if

$$kr_{\alpha} < l < \frac{kv}{\omega_{av\alpha}} = \frac{2E}{\hbar\omega_{\alpha}} \equiv l_2 \quad (18)$$

where  $\hbar\omega_{\alpha}$  is the average excitation energy of those excited states reached by the matrix element  $\langle n|X|0 \rangle^2$  which involve excitation of the orbital  $\alpha$ . It is reasonable to take

$$\hbar\omega_{\alpha} = \frac{\hbar^2}{2mr_{\alpha}^2} \quad (18a)$$

this relation is connected with the oscillator strength sum rule. Then the upper limit in (18) may be written

$$l_2 = 2k^2 r_{\alpha}^2 \leq 2k^2 a^2. \quad (19)$$

The lower limit in (18) comes from the fact that for  $b < a$ , the expansion of  $(|r_0 - r_i|)^{-1}$  in (7) is no longer justified. The transition probability  $1 - a_l^2$  is then smaller than (17); probably, a good estimate is to substitute  $kr_{\alpha}$  instead of  $l$  in (17).

Fortunately, the inelastic effect for  $b < a$  is relatively unimportant because there the phase shift is expected to be larger than  $1 - a_l$ . The WKB approximation gives

$$\delta_l = (\hbar v)^{-1} \int_{-\infty}^{\infty} V[(b^2 + z^2)^{\frac{1}{2}}] dz \quad (20)$$

where  $b = \lambda l$  and  $V(r)$  is the potential. Roughly, (20) gives

$$\delta_l \approx \frac{bV(b)}{\hbar v} = Z_P(b) \frac{e^2}{\hbar v} = Z_P(b)/ka_0 \quad (21)$$

where  $Z_P(r)$  is the "effective nuclear charge for the potential" used by Hartree. If the "atomic radius" is defined as  $r_1$ , the radius of the outermost shell, we expect approximately

$$Z_P(a) = \frac{1}{2}z_1 \quad (22)$$

where  $z_1$  is the number of electrons in this shell. Then

$$\delta(l = ka) \approx z_1/2ka_0 \quad (23)$$

On the other hand, if we take only the term  $\alpha = 1$  in (17); (i.e., only the outermost shell), then

$$1 - a(l = ka) \approx \frac{2}{3} \frac{z_1}{(ka_0)^2} \ll \delta(ka) \quad (24)$$

since we assume

$$ka_0 \gg 1. \quad (25)$$

Thus for  $b < a$ , the phase shift dominates, while for  $b > a$ , it goes rapidly to zero. We may then write (2) in the form

$$f = f_1 + f_2 + f_3 \quad (26)$$

$$f_1 = k^{-1} \sum_0^{ka} (2l+1) \sin \delta_l e^{i\delta_l} P_l(\cos \theta) \quad (26a)$$

$$f_2 = (i/2k) \sum_{ka}^{l_2} (1-a_l)(2l+1) P_l(\cos \theta) \quad (26b)$$

$$f_3 = (i/2k) \sum_0^{ka} (2l+1)(1-a_l) e^{2i\delta_l} P_l(\cos \theta). \quad (26c)$$

(26a) is the usual elastic scattering without absorption, (26b) is the absorptive effect due to distant collisions, and (26c) that due to close collisions. We shall show that  $f_3$  is unimportant.

We now insert (17) into (26b). Since  $ka \equiv l_1 \gg 1$ , we may neglect 1 compared with  $l$  and replace the sum by an integral, thus [8]

$$f_2 = \frac{iC}{k} \int_{l_1}^{l_2} \frac{l dl}{l^2} P_l(\cos \theta) \quad (27)$$



Since all  $l$ 's are large, this integral will be appreciable only for small  $\theta$ . It is therefore a good approximation to write

$$P_l(\cos \theta) = J_0(l\theta) \quad (28)$$

so that

$$f_2 = \frac{iC}{k} \int_{x_1}^{x_2} \frac{dx}{x} J_0(x) \quad (29)$$

$$x_1 = l_1 \theta = ka\theta \quad (29a)$$

$$x_2 = l_2 \theta = 2k^2 a^2 \theta. \quad (29b)$$

Since the energy is high, Eq. (25), we have:

$$x_2 \gg x_1. \quad (29c)$$

We have then 3 regions to consider

(a)  $x_1 \gg 1$ . In this case, the integral (29) is small. In other words, the absorptive scattering is small outside the diffraction region, i.e. for

$$\theta > 1/ka. \quad (30)$$

Thus we have a typical case of shadow scattering.

(b)  $x_1 \ll 1 \ll x_2$ . In this case, we may replace  $x_2$  by  $\infty$  in (29), the error being of order  $x_2^{-3/2}$ . Then

$$\int_{x_1}^{\infty} \frac{dx}{x} J_0(x) = -\ln ka\theta + 0.11593 + \int_0^{x_1} \frac{dx}{x} (1 - J_0). \quad (31)$$

The number 0.11593 represents  $\ln 2 - C$ , where  $C$  is Euler's constant. The last integral is very small.

(c)  $x_2 < 1$ . In this case, we may replace  $J_0(x)$  by 1, i.e. we have the constant, forward cross section,

$$f_2(0) = (iC/k) \ln x_2/x_1 = (iC/k) \ln 2ka. \quad (32)$$

In  $f_3$ , we assume, as previously discussed

$$1 - a_l = C/(ka)^2. \quad (33)$$

An upper limit will be obtained if we replace  $\delta_l$  in (26c) by 0. In cases (b) and (c), i.e. if  $x_1 \ll 1$ , we may replace  $P_l$  by 1. Then

$$-if_3 < C/2k \quad (34)$$

which amounts to adding  $\frac{1}{2}$  to (31) and to the log in (32). Thus  $f_3$  is indeed not very important. However, if (23) is small (Born approximation good), then the right hand side of (34) is a good estimate of  $-if_3$ .

We shall now investigate the *importance* of this "atomic shadow scattering". For this purpose, we consider the imaginary part of the forward scattering amplitude which is given by the optical theorem

$$\text{Im } f(0) = (k/4\pi)\sigma_{\text{tot}}. \quad (35)$$

We shall therefore compare the total cross sections for elastic and inelastic scattering.

The total inelastic scattering cross section is from (4), (17), (18) and (33)

$$\begin{aligned} \sigma_{\text{inel, tot}} &= \frac{\pi}{k^2} 2C \left[ \sum_{ka}^{l_2} \frac{(2l+1)}{l^2} + \sum_0^{ka} \frac{(2l+1)}{(ka)^2} \right] \\ &\approx \frac{4\pi}{k^2} C \left( \ln \frac{l_2}{ka} + \frac{1}{2} \right) \\ &= \frac{4\pi}{k^2} C (\ln 2ka + \frac{1}{2}). \end{aligned} \quad (36)$$

Inserting  $C$  from (17), and replacing  $a$  inside the  $\ln$  by  $r_\alpha$ , as we should according to (18), (19), we get

$$\sigma_{\text{inel, tot}} \approx \frac{8\pi}{3k^2 a_0^2} \sum_\alpha z_\alpha r_\alpha^2 (\ln 2kr_\alpha + \frac{1}{2}). \quad (37)$$

The total elastic scattering is somewhat more difficult. In the Born approximation, we have the well-known formula [9]

$$\sigma_{\text{el}} = \frac{2\pi}{k^2} \int_0^{2k} q dq \frac{4}{a_0^2 q^4} (Z - F(q))^2 \quad (38)$$

where the form factor is given by

$$F(q) = \int \rho(r) \frac{\sin qr}{qr} d\tau. \quad (39)$$

With our assumption of separate electron shells  $\alpha$ , we have approximately

$$Z - F(q) = \sum_{\alpha} z_{\alpha} \left( 1 - \frac{\sin qr_{\alpha}}{qr_{\alpha}} \right). \quad (40)$$

For any given  $q$ , usually one of the shells  $\alpha$  dominates, viz. that for which  $qr_{\alpha} \approx 1$  to 3. Accordingly, we evaluate (38) by adding the squares of the terms  $\alpha$  in (40), leaving out the mixed terms; this underestimates (38), but probably not greatly. Then the integral in (38) can be evaluated and we get

$$\sigma_{\text{el}} = \frac{8\pi}{3k^2 a_0^2} \sum_{\alpha} z_{\alpha}^2 r_{\alpha}^2 (\ln 2 + \frac{1}{12}). \quad (41)$$

The outermost electrons ( $\alpha = 1, r_{\alpha} = a$ ) give the main contribution, both here and in (37).

For any given  $l$ , or given impact parameter  $b = \lambda l$ , the Born approximation is fairly good as long as the phase shift is less than one. According to (21), this means for a given  $b$

$$Z_p(b) \equiv bV(b) < ka_0. \quad (42)$$

For (41) to be a good approximation, it is necessary that (42) be fulfilled for  $b = r_1 = a$ : then the contribution of the outermost shell,  $\alpha = 1$ , to (41) is correctly given, and those of the inner shells are small by comparison. Therefore we must have

$$Z_p(a) < ka_0. \quad (42a)$$

If (42a) is not fulfilled, a rough approximation is obtained by the assumption that all phase shifts up to  $l = ka$  are large; then  $\sin^2 \delta$  is on the average  $\frac{1}{2}$  for  $l < ka$ , and we obtain

$$\sigma_{\text{el}} \approx 2\pi a^2. \quad (43)$$

This includes the shadow scattering due to the elastic collisions. An alternative criterion for the Born approximation is that (41) should be less than (43) which requires

$$ka_0 > z_1 \quad (44)$$

a condition very similar to (42a).

Taking just the shell of largest radius,  $\alpha = 1$ , in (41) and (37), we get

$$\frac{\sigma_{\text{inel}}}{\sigma_{\text{el}}} = \frac{1.29}{z_1} (\ln 2ka + \frac{1}{2}). \quad (45)$$

The ratio of inelastic to elastic scattering increases slowly with energy. However, it is never very large unless the number of electrons in the outermost shell,  $z_1$ , is very small. Therefore the shadow scattering discussed in this paper should be most important for H and He. Indeed it is for He that Massey and Mohr noticed the large forward scattering, and interpreted it in terms of their imaginary potential, i.e. as shadow scattering.

Our theory is particularly simple for H and He. These atoms have only a single electron shell, so that the estimates (41) for the elastic scattering, and (45) for the ratio, are both good (better expressions could easily be obtained for both, using explicit wave functions for the atomic electrons). Moreover, the Born approximation is valid already for very low electron energy. Then the contribution  $f_1$  in (26a) is purely real (ordinary elastic scattering), while  $f_2$  and  $f_3$  are purely imaginary (absorptive scattering). The cross section is simply

$$\frac{d\sigma}{d\Omega} = f_1^2 + |f_2 + f_3|^2 \quad (46)$$

The forward peak due to  $f_2 + f_3$  appears in its purest form. For the quantitative results, see Massey and Mohr [4].

For alkalis and similar atoms, the contributions of the *two* outer shells must be taken into account in (37), (41) because the next-to-outer shell has many more electrons (8) than the outermost one. For noble gases other than helium,  $z_1$  is large ( $= 8$ ), so  $\sigma_{\text{inel}} \lesssim \sigma_{\text{el}}$ . Thus, for experimentally important energies, the forward scattering is *mostly* due to the elastic total cross section. Nevertheless, the narrow peak, of angular width  $1/ka$ , which represents the "shadow" of the inelastic scattering, should still be noticeable over the smoother background of the "purely elastic" scattering  $|f_1|^2$ .

A similar effect should exist in the scattering of electrons by nuclei. The excitation of the giant resonance is a dipole interaction which can therefore occur for relatively distant collisions of the electron. This also should give rise to a sharply forward-peaked shadow. However, because of the strong direct Coulomb scattering, this is probably difficult to observe.



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# C VIOLATION IN STRONG INTERACTIONS

J. PRENTKI

*CERN, Geneva and Collège de France, Paris*

M. VELTMAN

*CERN, Geneva*

*(Received May 28, 1965)*

Once more physicists are facing the breakdown of a principle, namely  $CP$  invariance, which was supposed to be generally valid. As is the hallmark of a general principle, the consequences are simple to understand on the one hand, but far reaching on the other. It is just these features which make the subject so attractive for study both theoretically and experimentally; we are therefore happy to dedicate this account to Professor V. F. Weisskopf, whose deep interest in such matters, and whose warm personality and high standards in science and scientific life have made such an impact on CERN.

In this discussion we will limit ourselves to the question of possible  $C$  violation in strong interactions and where it may manifest itself in an observable manner. As yet it is still an open question whether the  $CP$  violation observed in  $K_L \rightarrow 2\pi$  decay [1] is due to a perturbation by a rather strong interaction [2]; moreover, even if we assume that such is the case, we are still in doubt about number of properties of this interaction. The first question that arises is: does there exist a class of interactions to which we can attribute this  $CP$  violation? To this purpose we must first establish what we understand under a "class of interactions". Three notions are important in this respect, namely symmetry properties, the involved particles, and strength, and interaction classes are distinguished from each other through behaviour with respect to one or more of these qualifications. Thus, a class of interactions may distinguish itself from another class through different behaviour with respect to some symmetry; a good example is the so-called medium strong or  $SU_3$  breaking interaction whose existence became significant only after the discovery of  $SU_3$  symmetry. These

$SU_3$  breaking interactions are not yet sharply defined through a strength of coupling constant; this is in contrast to the weak interactions that always have been characterized by their small strength, whether they are leptonic, non-leptonic, strangeness violating or strangeness conserving. Thus, when parity violation was used to explain the famous  $\theta-\tau$  puzzle, without any further ado this parity violation was generalized to all weak interactions. The electromagnetic interactions are characterized through the photon being involved, and also by their strength and behaviour with respect to isospin, which is used as identification if the photon is only virtually present. Indeed, it would be very difficult to distinguish an  $I$  spin breaking interaction among strongly interacting particles of a strength of about  $10^{-2}$ , and unless some further observable difference is detected it remains a question of semantics.

Let us now investigate what properties we reasonably can attribute to the  $C$  violating interaction. First we must discuss somewhat in detail the  $K$  meson system. One knows that the  $\Delta I = \frac{1}{2}$  selection rule is broken in  $K^+ - K^0 \rightarrow 2\pi$  decay, a 5 %  $\Delta I = \frac{3}{2}$  or  $\frac{5}{2}$  amplitude admixture being observed. This is a somewhat stronger breaking than expected from electromagnetism, but there are some arguments that explain this discrepancy [3]. The possibility that these  $\Delta I = \frac{3}{2}$  or  $\frac{5}{2}$  amplitudes arise from a  $C$  violating,  $I$  spin breaking perturbing interaction [4] is not very plausible, because one would expect in such a case a 5 % effect in  $K_L \rightarrow 2\pi$  instead of the observed 0.25 % [5]. Thus, we will continue to assume the  $\Delta I = \frac{3}{2}$  or  $\frac{5}{2}$  amplitudes as resulting from e.m. perturbations, that conserve  $C$ . If the  $C$  violating interaction breaks  $I$  spin also it must thus be of strength  $10^{-2}-10^{-3}$ , and give rise to a  $\Delta I = \frac{3}{2}$  or  $\frac{5}{2}$  amplitude small with respect to the 5 % electromagnetic amplitudes. Although this is not experimentally excluded it is clearly more attractive to assume that the  $C$  violating interactions conserve  $I$  spin. This lifts also somewhat the restriction on the magnitude of the coupling constant [6].

Thus we assume the following selection rules:

- 1)  $\Delta I = 0$ ;
- 2) parity conservation. Parity non-conserving effects seem to appear only at the level of weak interactions. An example of a test is the absence of an electric dipole moment for the neutron [7];

3)  $\Delta S = 0$ . A glance in any table on properties of elementary particles shows that strangeness is broken only at the level of weak interactions;

4) obviously, to be able to act in the  $K^0$  system the interaction must involve strongly interacting particles.

Altogether we have an interaction between strongly interacting particles with strength  $\gtrsim 10^{-2}$ ,  $\Delta I = 0$ ,  $\Delta S = 0$ , and  $P$  conservation. Within the present possibilities for distinguishing classes of interactions we arrive at the conclusion that we are dealing with strong interactions that may or may not break  $SU_3$ .

From some general considerations we may further arrive at certain limitations. On the basis of a simple theorem due to Soloviev, Pais and others one may find it plausible that no  $C$  violation occurs in the  $SU_3$  conserving interactions. Recently this point has been analyzed anew by Cabibbo [8], who has been able to state a number of theorems of this nature for matrix elements rather than for interaction Lagrangians, and it appears reasonable that in many cases  $C$  violation even in strong interactions only shows up at the level of  $SU_3$  breaking interactions. For the time being the  $SU_3$  behaviour of a  $C$  violating interaction is quite academic, but ultimately (if indeed the  $C$  violation is to be found in the strong interactions) this question must be settled.

In the following we will concentrate our attention on the detection of a  $\Delta I = 0$ ,  $C$  violating interaction, occasionally mentioning tests for  $\Delta I \neq 0$ ,  $C$  violating interactions. Let us discuss some interesting reactions. The ideal systems for direct observation are those systems that are eigenstates of  $C$ , and we will limit ourselves here to such systems, excluding discussion of possible  $C$  violating effects, for example, in collision processes.

From the table of elementary particles and resonances [9] we find as candidates (with the exclusion of  $K_1^0$  or  $K_2^0$  and some doubtful cases):

$$\pi^0, \eta, X^0 (= \eta 2\pi), \rho^0, \omega, \phi.$$

Further we have the (by far the most interesting) proton-antiproton system.

$\pi^0$  decay. The only particles lighter than the pion are leptons or



photons. The  $C$  violating decays are:

$$\begin{aligned}\pi^0 &\rightarrow (\gamma) \rightarrow e^+ + e^- \\ \pi^0 &\rightarrow 3\gamma, \text{ etc.}\end{aligned}$$

The first process is forbidden in lowest order of electromagnetic interactions because of parity, and also gauge invariance. Of course, this transition may proceed in higher order, see the figure. The process  $\pi^0 \rightarrow 3\gamma$  has not yet been looked for with an interesting accuracy. An estimate of the rate on the basis of simple phenomenological considerations has been made by Berends [10], and the conclusion is that this process is probably very rare [11].

$\eta$  decay. The  $\eta$  decay into  $\pi^+\pi^-\pi^0$  offers the extremely interesting possibility of an interference between an electromagnetically induced and a  $C$  violating transition. As has been noted however [12] angular momentum barrier effects play a very important role here, and it is not easy to estimate possible effects. The decay modes in question and their estimated strength are ( $e^2 = 1/137 = \text{e.m. coupling constant}$ ):

Mode	$I$ spin viol.	Strength	$C$ behaviour	Wave function
0	$\Delta I = 0$	$gk^3$ or $g'e^2k^3$	$C$ viol.	$\partial_{\mu\nu\alpha}^3 \eta e^{ijk} \partial_{\mu\nu} \pi^i \partial_\alpha \pi^j \pi^k$
1	$\Delta I = 1$	$e^2$ [13]	$C$ cons.	$\eta(\pi^i \pi^i) \pi^j$
2	$\Delta I = 2$	$ge^2k$ or $g'k$	$C$ viol.	$\partial_\mu \eta \partial_\mu \pi^i \pi^j \pi^l e^{ijk}$

Only the wave functions with minimum of derivatives for a given  $I$  spin mode are considered. The uninteresting  $\Delta I = 3$  mode has been left out.  $\partial_\mu$  stands for  $\partial/\partial x_\mu$ .

In here  $g$  is the coupling constant of the  $C$  violating isospin conserving interactions; for completeness we added also the case that the  $C$  violating interaction breaks  $I$  spin also and called that coupling constant  $g'$ . Latin indices indicate  $I$  spin components.  $k$  represents angular momentum barrier effects:

$$k = \frac{m_\eta Q}{M^2}$$

where  $Q$  is the average kinetic energy of the pions, about 50 MeV, and  $M$  is some unknown reference mass, certainly larger than the mass of the pion.  $m_\eta$  is the  $\eta$  mass.

As has been pointed out [14] interference between a  $C$  conserving and a  $C$  violating mode may result in that the ratio

$$\frac{\text{Number of events with } \pi^+ \text{ energy} > \pi^- \text{ energy}}{\text{Number of events with } \pi^+ \text{ energy} < \pi^- \text{ energy}}$$

is different from 1. Study of the Dalitz plot may eventually reveal whether mode 0 or 2 is the interfering one. It may be noted that the known dominant  $S$  wave structure of the Dalitz plot implies the dominance of mode 1.

Let us write down the energy dependence of the matrix element for the different modes. Denoting the energy (including rest mass) of  $\pi^+$ ,  $\pi^-$  and  $\pi^0$  by  $E_+$ ,  $E_-$  and  $E_0$  we have:

Mode	Energy dependence of matrix element
0	$E_\eta^3 \{ E_0^2(E_- - E_+) + E_+^2(E_0 - E_-) + E_-^2(E_+ - E_0) \}$ $= \frac{1}{4}x(x^2 - 3y^2) \cdot E_\eta^3$
1	Const.
2	$E_\eta(E_+ - E_-) = E_\eta x$

where  $x = E_+ - E_-$ ,  $y = E_0 - \frac{1}{3}m_\eta$ . The mode 0 matrix element is antisymmetrical between the three pions. Neither mode 0 nor mode 2 give rise to the decay  $\eta \rightarrow 3\pi^0$  [15].

Another  $\eta$  decay mode is  $\eta \rightarrow \pi^+\pi^-\gamma$ . In this decay  $C$  violating interactions can interfere, but they would suffer quite strong angular momentum barrier effects. Moreover, the  $C$  violating modes are suppressed by an extra factor  $g$  as compared to the main electromagnetic mode.

$X^0$  decay. As the  $X^0$  has the same quantum numbers as the  $\eta$ , everything said above is applicable to  $X^0$  decay also. Thus the  $3\pi$  mode (not observed yet) is of particular interest, especially because barrier effects should be less important. Here we have the drawback of a competing strong process, namely  $X^0 \rightarrow \eta\pi\pi$ . The decay  $X^0 \rightarrow \eta\pi\pi$  is  $G$  parity conserving and any interfering  $C$  violation must break isospin also. Thus this decay is suitable for detection of  $C$  violating interactions with  $\Delta I = 1$ .

$\rho$  decay [16].  $\rho^0 \rightarrow \eta\pi^0$  is forbidden if  $C$  is conserved.  $\rho^\pm \rightarrow \eta\pi^\pm$  may proceed through  $C$  violation or (electromagnetically)  $I$  spin

violation. The branching ratio expected for the  $C$  violating case could be at most  $g^2 \lesssim 10^{-2}$  with respect to the main mode  $\rho \rightarrow 2\pi$ , the e.m. decay should be down by a factor  $e^4 \sim 10^{-4}$ . It is interesting to note that  $\rho^0 \rightarrow \eta\pi^0$  could simulate a resonance of  $I$  spin 0 in the  $\rho$  region.

$\omega$  and  $\phi$  decay [16]. The decays  $\omega \rightarrow \eta\pi$ ,  $\omega \rightarrow 3\pi$  and  $\phi \rightarrow \eta\pi$ ,  $\phi \rightarrow 3\pi$  conserve  $G$  parity and can therefore be used only to detect  $C$  and  $I$  spin violating interactions. Note that the normal  $\phi \rightarrow 3\pi$  is strongly suppressed (by  $SU_6$ ) so that any irregular decay could show up stronger. Barrier effects are very important here, too.

Very interesting are  $\omega \rightarrow \pi\pi\gamma$  and  $\phi \rightarrow \pi\pi\gamma$ . Depending on the pion configuration  $C$  is violated or conserved (as  $\Delta I = 0$  or 1 there is no limitation from isospin). Thus interference may show up as asymmetries between the  $\pi^+$  and  $\pi^-$  distributions. A favourable circumstance is the possible enhancement of the  $C$  violating mode through the  $\rho$  meson:  $\omega \rightarrow \rho\gamma$  ( $\phi \rightarrow \rho\gamma$ ) is forbidden by  $C$ . As has been noted elsewhere [17] this decay may be used for completely different purposes, namely the detection of  $S$  wave  $\pi\pi$  resonances.

The  $p\bar{p}$  system [18]. The above-mentioned decay modes may all be used to detect the existence of  $C$  violations, and eventually we could get information on  $I$  spin behaviour and also on the strength of the  $C$  violating interaction. But as no strange particles are involved it is not easy to see how information with respect to  $SU_3$  could be obtained. For these purposes the  $p\bar{p}$  system is well suited:  $K$  and  $K^*$  mesons are quite copiously produced and if  $C$  violation shows up here a systematic study could reveal properties with respect to  $SU_3$ . In this context also tests of the kind as proposed in Ref. [19] ( $p\bar{p} \rightarrow A\bar{A}$ ) may provide very useful information.

In the following we will not try to give a general discussion. We merely note the following interesting fact: if  $C$  is conserved the energy spectra and total numbers of  $K^+$  and  $K^-$  in the reactions

$$p + \bar{p} \rightarrow K^\pm + \text{anything}$$

must be identical (in the  $p\bar{p}$  centre-of-mass system). If  $C$  is not conserved this need not be the case, which we will demonstrate on a simple example, namely the channel:

$$p + \bar{p} \rightarrow K\bar{K}\pi^\pm$$

with  $p\bar{p}$  annihilation at rest. There are two reasons why we take this channel: first, it has been demonstrated [20] that with certain assumptions  $\tilde{U}(12)$ , one of the relativistic generalizations of  $SU_6$  forbids this transition, which we interpret that it could be that the  $SU_3$  invariant transitions are somewhat suppressed so that other effects may show up more easily; and second, because the isotopic spin structure is very simple, which saves us writing. For a proton and an antiproton at rest in a state of zero angular momentum the only non-zero spinor combinations are

$$\bar{u}_p \gamma^\mu u_p \text{ and } \bar{u}_p \gamma^5 u_p$$

i.e., the  $^3S_1$  and  $^1S_0$  state, respectively. Both have the parity minus, the  $^3S_1$  state has  $C = -1$  (like e.m. current), the  $^1S_0$  state has  $C = +1$ . We will limit ourselves to the  $^1S_0$  state.

With respect to isotopic spin the  $p\bar{p}$  system is an equal mixture of isotopic spin 0 and 1

$$(p\bar{p}) = \frac{1}{2}\{(\bar{N}N) + (\bar{N}\tau_3 N)\}.$$

Thus the  $^1S_0$  state contains an equal mixture of states with  $I$  spin 0 and 1, both with parity  $-$  and  $C +$ , i.e., of  $\eta$  and  $\pi^0$  like states. Thus the  $^1S_0$  state is a mixture of two states with different  $G$  parity. Clearly then any final state with a definite  $G$  parity may be reached by both the  $G$  conserving and the  $G$  violating ( $= C$  violating if  $I$  spin is conserved) interactions, and interference effects may show up.

Another possibility is that two final states, with different  $G$  parity, are reached from the same initial state by  $C$  violating and  $C$  conserving interactions and interfere in an observable way. As a first example we consider the case where the  $K$  mesons are in an  $S$  wave with respect to each other. As we consider only the  $\bar{K}^0 K^+$  and  $K^0 K^-$  combinations only the  $I$  spin 1 combination of the kaons is important, and this combination has the  $G$  parity minus (being an isovector with  $C = +1$ ). Together with the pion we have a system with  $G = +1$ , and the  $C$  conserving (violating) transition proceeds from the  $I$  spin 0 (1) state of  $p\bar{p}$ . The general matrix element is

$$M_1 = a\{(\bar{N}N)\pi^i(\bar{K}\tau^i K)\} + b\{\bar{N}\tau^i N\}\pi^j(\bar{K}\tau^j K)\epsilon^{ijk}\}$$

where we indicated only the isospin structure. If  $b$  is non-zero  $C$  is



violated. In the absence of final state interactions  $a$  and  $b$  have the same phase (are real), but the final state interactions destroy this property. The ratio of  $p\bar{p} \rightarrow K^+ \bar{K}^0 \pi^-$  to  $p\bar{p} \rightarrow K^- \bar{K}^0 \pi^+$  is given by

$$\frac{\text{rate}(K^+ \bar{K}^0 \pi^-)}{\text{rate}(K^- \bar{K}^0 \pi^+)} = \left| \frac{a+ib}{a-ib} \right|^2 = \frac{|a|^2 + |b|^2 - 2\text{Im}(ab^*)}{|a|^2 + |b|^2 + 2\text{Im}(ab^*)}$$

which is not necessarily 1.

To demonstrate the other class of interference phenomena in this particular channel we may consider interference between systems where the  $K$  mesons are in an  $S$  or  $P$  wave, respectively. The latter combination has the  $G$  parity plus, and the general matrix element is the sum of the  $S$  wave matrix element  $M_1$  above and the  $P$  wave matrix element  $M_2$

$$M = M_1 + M_2$$

$$M_2 = a' \{(\bar{N}N) \partial_\mu \pi^i (\bar{K} \overset{\leftrightarrow}{\partial}_\mu \tau^i K)\} + b' \{(\bar{N} \tau^i N) \partial_\mu \pi^j (\bar{K} \overset{\leftrightarrow}{\partial}_\mu \tau^k K) \epsilon^{ijk}\}$$

where

$$(\bar{K} \overset{\leftrightarrow}{\partial}_\mu \tau^i K) = (\partial_\mu \bar{K}) \tau^i K - \bar{K} \tau^i (\partial_\mu K).$$

In  $M_2$  we have  $C$  violation if  $a' = 0$ . Interference between  $a'$  and  $a$  may give rise to different energy spectra for  $K^+$  and  $K^-$ , but after integration over all energies these interferences drop out and total numbers of  $K^+$  and  $K^-$  are not influenced by this effect. This type of interferences may be more easily accessible to detection if for some reason the  $C$  violating transition is enhanced or the  $C$  conserving one depressed (angular momentum barriers, resonant states, etc.).

Thus we arrive at the conclusion: if the energy spectra and total numbers of  $K^+$  and  $K^-$  are different for any one channel  $C$  is violated.

Obviously similar statements can be made with respect to pions or resonant states instead of kaons.

#### ACKNOWLEDGEMENTS

The authors are indebted to Drs. J. S. Bell and N. Cabibbo and Professor L. Van Hove for encouragement and stimulating discussions.

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A number of experiments of the type
 
$$\alpha + C \rightleftharpoons d + N$$
 has been performed, and any violation of detailed balance there is below about 3 %. As these reactions are governed by  $SU_3$  conserving forces and moreover do not involve strange particles we do not expect big effects. See also Ref. [8];  
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As the mass matrix governing the definition of  $K_L$  and  $K_S$  will only be affected by a few %, there will be no compensation between a phase in the mass matrix and the  $\frac{3}{2}$  or  $\frac{5}{2}$  amplitude phase (which can in these circumstances be anything between  $0^\circ$  and  $360^\circ$ ). In other words,  $K_L$  will, up to a few percent, still be eigenstate of  $CP$ , but we can say nothing of the  $\frac{3}{2}$  or  $\frac{5}{2}$  amplitudes. See also Ref. [6].
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In this case the  $\Delta I = \frac{3}{2}$  and  $\frac{5}{2}$  transitions suffer only small perturbations, of a few percent, from the  $C$  violating interaction, and their phase with respect to the mass matrix will be close to zero. The main contribution to  $K_L \rightarrow 2\pi$  should come from the  $\Delta I = \frac{1}{2}$  transition, being out of phase with the mass matrix by a small amount. If the  $\Delta I = \frac{1}{2}$  mode is the main constituent of the mass matrix, the phase of the latter may be very close to the  $\Delta I = \frac{1}{2}$  amplitude phase.
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From  $SU_3$  one may obtain an estimate of the total rate  $\eta \rightarrow 2\gamma$  using the rate  $\pi^0 \rightarrow 2\gamma$  as input (result:  $\tau^{-1} \sim 140$  eV). As  $\eta \rightarrow 3\pi$  is about just as abundant as  $\eta \rightarrow 2\gamma$  one can then calculate the coupling constant  $f$  involved for an S wave decay (mode 1). The result is larger than expected from electromagnetism:  $f^2/4\pi = 2.5 \cdot 10^{-3}$  instead of  $e^4 \sim 10^{-4}$ . It seems that this enhancement is a common feature of virtual e.m. processes, as well as weak processes (the non-leptonic are generally a factor 10 stronger than leptonic ones).
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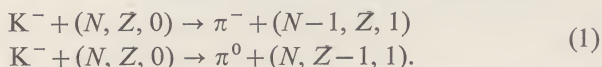
# STUDIES OF HYPERNUCLEI WITH K MESON BEAMS \*

H. FESHBACH and A. K. KERMAN

*Department of Physics and Laboratory for Nuclear Science  
Massachusetts Institute of Technology, Cambridge, Massachusetts*

*(Received June 2, 1965)*

In the not too distant future, we can look forward to the possibility of K meson beams with sufficient intensity to do precision measurements on the production of hypernuclei in the collision of a kaon with a complex nucleus. Some of the possible nuclear and elementary particle information obtainable from hypernuclei properties were discussed by D. H. Wilkinson and R. Dalitz in two conferences held at CERN early in 1963 [1, 2]. In this note we shall be particularly interested in those hypercharge exchange reactions in which the final state consists of a hypernucleus and a  $\pi$  or K meson. For example, single hypernuclei can be produced in the reactions \*\*.



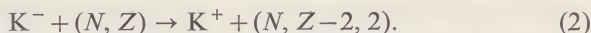
It is important to realize that by choosing the appropriate energy for the kaons, it is possible to minimize the momentum transfer which the target nucleus must absorb. For example, if the kaon has an energy of about 210 MeV, the collision between a kaon and a nucleon at rest will, in the forward direction, result in a  $\Lambda$  hyperon at rest and an energetic pion. In the neighborhood of this energy, the form factor involved in reaction (1) will have its maximum value, and it becomes very likely that the  $\Lambda$  hyperon will be trapped in a bound state. In contrast to this K<sup>-</sup> mesons captured at rest give a momentum of  $1\text{f}^{-1}$  to the  $\Lambda$ , so that it becomes more probable that the final state will be a star rather than a two-body system as in reactions (1).

\* This work is supported in part through funds provided by the Atomic Energy Commission under Contract AT(30-1)-2098.

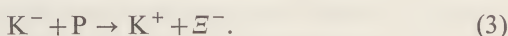
\*\* The notation  $(N, Z, n)$  represents a nucleus with  $N$  neutrons,  $Z$  protons,  $n\Lambda$  hyperons.



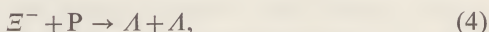
Among the reactions which will be weaker because of the larger momentum transfer is the intriguing possibility in which a doubly strange hypernucleus is formed:



This reaction involves at least a double scattering with two of the nucleons in the target nucleus. For example, in the first scattering a cascade particle can be produced



Since the mass of a proton plus a cascade exceeds the mass of two  $\Lambda$ 's, one can expect \* that the  $\Xi^-$  will scatter with a proton to finally produce two  $\Lambda$ 's:



This state can also be produced by two successive interactions in which a  $\pi^0$  created in the first collision of a kaon with a nucleon in the target nucleus interacts with another target nucleon to produce another  $\Lambda$  hyperon and a  $K^+$ . In either event it is clear that the target nucleus will undergo a considerable momentum transfer. The probability for the production of double hypernuclei via reaction (2) should therefore be considerably smaller than the probability for single hypernucleus production via reaction (1).

In both reactions (1) and (2), it is possible to demonstrate the existence of the hypernucleus and to determine its energy spectrum by measuring the momentum (magnitude and direction) of the emerging meson. If such experiments could be performed, they would be useful for both light and heavy hypernuclei, since the hypernucleus is not detected as is presently the case by its decay, and one no longer need require that the mesonic decays be a detectable fraction of all the decays.

What special features of the energy spectrum of a single hypernucleus are of interest? In a single hypernucleus, one might try to distinguish the excited states formed by the promotion of a  $\Lambda$  to excited

\* Since this mass difference is comparatively small ( $\sim 28$  MeV), it is possible though not likely that states exist in which the  $(\Xi^-, \text{nucleus})$  system is stable against decay into the  $\Lambda^2$  configuration.

single particle levels from those arising from nucleon excitation. The former would be expected to have a characteristic angular distribution, since it can occur in a direct reaction, whereas nucleon excitation requires a more complex process. One can also envisage that the nuclear core associated with each single particle hyperonic state might exist in a number of collective excited states.

In nuclei above Be the  $\Lambda$  is bound by more than 8 MeV. As the energy of excitation of the  $\Lambda$  increases above roughly 8 MeV, we enter a region in which the hypernucleus is unstable against neutron emission. The  $\pi$  meson spectrum resulting from the formation of the hypernucleus, which below this energy is, in principle, discrete, now becomes a continuum. However, one can expect sharp peaks in the cross-section. Their energies would roughly be at those values where the hypernucleus would have a bound state if the neutron channel were not open. Of course these energies will be shifted by the coupling to the continuum. The peaks will have a width which will depend upon the neutron energy of emission and the coupling between the excited  $\Lambda$ -nucleus channel and the open channel. Indeed, depending upon the nature of this coupling, one may find narrow widths for these resonances very much like those which exist for neutrons incident on ordinary nuclei. These may never be detectable with the kind of resolution conceivable at 200 MeV, but the strength function associated with single particle states of the  $\Lambda$  might very well be visible with 1 MeV resolution.

For the case of double hypernuclei, much could be learned about the forces between two  $\Lambda$ 's in the presence of the nuclear core, especially if the spectrum of single  $\Lambda$  excited states were known from the single hypernuclei. Double hyper-fragments could provide a test of statistics of the  $\Lambda$  hyperons, since the spin of the ground states of double hypernuclei would be sensitive to this feature.

A particularly interesting subject to consider is the possibility of hyperonic analogue states similar to the isobaric analogue states recently found in ordinary nuclei [3]. Isobaric analogue states are formed from the nuclear ground (or a low lying) state by substituting a proton for one of the neutrons in such a way as not to change the space-spin wave function. Such states are never \* the ground state of the nucleus

\* Except for the special case of  $I = \frac{1}{2}$ .

so formed, since the latter has a different symmetry, but they have been found to be extremely pure, indicating that isobaric spin is a good quantum number in spite of the action of the symmetry breaking coulomb force.

We envisage (similar considerations have been put forward by L. S. Kisslinger, private communication) the formation of a hyperonic analogue state by replacing one neutron in any nucleus by a  $\Lambda$  in such a way as not to change the space-spin wave function. One might expect that such a state of the single hypernucleus (of course one can suggest analog states in the double hypernuclei also!) would be preferentially excited by reaction (1) because of the similarity between the initial and final spin-space wave function and so produce a peak in the cross sections. The width of such a peak and its very existence provide a test of the extent to which baryon wave functions possesses a certain symmetry. According to present-day thinking that symmetry is presumed to be that of  $SU_3$ , although the existence of the hyperonic analogue state is not inconsistent with any symmetry which considers the proton, neutron and the lambda hyperon to be initially degenerate states of a single particle. It may very well be that the existence of hyperonic analogue states is a very telling test of such symmetry in the baryon-baryon interaction. Thus, it is well to remember that the light nuclei and the existence, energy and width of the isobar analogue states provide the best evidence for the charge independence of nuclear forces.

If  $SU_3$  symmetry were perfect, the ground state of each of the ordinary nuclei would be associated with a definite representation determined by  $N$  and  $Z$  and the fact that its hyperonic charge is  $Y = (N+Z)$  and its isotopic spin  $I = \frac{1}{2}(N-Z)$ . The dimensionality of the representation is

$$D(N, Z) = (1+N-Z)(1+N+2Z)(1+N+\frac{1}{2}Z)$$

and the nucleus is at the top left hand corner of the isobaric spin, hypercharge diagram. The analog state of the single hyperfragment is supposed to be associated with the second row of the diagram which has hypercharge  $Y' = Y-1$  and isobaric spins of  $I' = I \pm \frac{1}{2}$ . Because of the large  $\Lambda$ ,  $\Sigma^0$  mass splitting (on the nuclear scale) we can only have the  $\Lambda$  in our single hypernucleus, so that it cannot be pure  $D(N, Z)$  but must be a mixture of representations with the same per-



mutation symmetry. This symmetry is different from that of the ground state of the single hyperfragment, since the  $\Lambda$  has no Pauli forbidden states and drops down to the bottom of its average well to form the ground state. We know that this well is the order of 25–30 MeV deep, because the  $\Lambda$  binding in heavy nuclei seems to level off as a function of  $A$  to about 20 MeV.

If  $SU_3$  symmetry were perfect, the binding of the hyperonic analogue state would be the same as that of the nuclear state from which it was formed, i.e. about 8 MeV. However, it is not perfect. We have already seen in the above paragraph that because of the difference between the  $\Lambda$  and  $\Sigma$  masses that the hyperonic analogue state involves a mixture of representations of  $SU_3$ . The shift in energy, i.e. the mass splitting, and the width of the analogue state depends upon the strength and range of symmetry breaking forces which originate in the difference between the pion and kaon mass. This means that the long range part of the baryon-baryon potential, the OPEP part of the nucleon-nucleon potential is symmetry breaking. An indication of its magnitude is given by the comparison between  $He^4$  and its hyperonic analogue  $_{\Lambda}He^4$ .  $He^4$  has a binding energy of 20 MeV,  $_{\Lambda}He^4$  only 2 MeV. Thus the symmetry breaking potential gives rise in this case \* to a mass splitting of about 20 MeV. If this value is maintained in the heavier nuclei, the hyperonic analogue would most probably be in the continuum several tens of MeV above the ground state. The width will be determined by the non-diagonal matrix elements of the symmetry breaking force. We estimate that the resultant width will be of the order of nucleon energies, i.e. of the order of one MeV.

In the experiments suggested above, the  $\Lambda$  hyperon bound in the hypernucleus probes the properties of the nuclear core. It differs from other probes in that it has the baryon mass, the baryon spin, and the baryon strong interaction, but since it is not a nucleon need not satisfy the Pauli principle. The energy spectrum of the hypernucleus, the energy and width of the hyperonic analogue states if they exist as a function of mass number would furnish data from which the  $\Lambda$ -nucleus interaction could be abstracted and from which some of the properties of the nuclear core with which the  $\Lambda$  interacts can be determined.

\* The fact that  $_{\Lambda}He^4$  is nearly not bound will cause an additional shift in the energy similar to the Thomas-Ehrman shift for the isobar analogue case.



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# ON THE QUANTUM THEORY OF ELECTRIC CONDUCTIVITY

W. THIRRING

*Institute for theoretical Physics \*, University of Vienna*

*(Received June 2, 1965)*

## 1. INTRODUCTION

In the standard textbooks [1] the theory of conductivity is usually developed from the Boltzmann equation. This is somewhat irritating since we know that electrons obey the Schrödinger equation and it is not trivial when the latter implies the former \*\*. A more satisfactory approach has been initiated by Kubo [2] where a formal expression for the conductivity  $\sigma$  is deduced from the Schrödinger equation. Unfortunately the exact evaluation of  $\sigma$  requires a solution of the manybody problem for which in general only approximation methods of unknown validity are available. It is the purpose of this note to investigate two systems which are simple enough so that they admit a complete mathematical treatment and yet show the relevant features, in particular a finite d.c. conductivity. The simplification consists in neglecting the interaction between the electrons which is not the relevant factor for the conductivity. In this way the manybody problem is reduced to a one-electron problem. One might worry whether the statistics of the electrons may introduce an essential complication because of the exclusion principle. However, it turns out that the conductivity of several electrons without mutual interaction is just the sum of the conductivities in the occupied states. One also finds that the thermodynamic complications are not the pertinent feature of the problem. Indeed it is possible to define the conductivity of a single electron in a quantum mechanical state. For a thermal ensemble the conductivity is a weighted sum of these. By restricting ourselves to the one-electron systems we disregard certain problems. For a homogeneous isotropic system the linear response to an arbitrary electromagnetic

\* This work was performed as consultant to General Atomics Europe.

\*\* A recent text on this question is Kadanov and Baym, *Quantum Statistical Mechanics* (W. A. Benjamin Inc., 1962).

field is expressible by two complex functions of frequency and wave number [3]. They are the dielectric constant and the magnetic permeability. We shall study only the former in dependence on the frequency for infinite wavelength. Although there is some simplification in the zero frequency limit it is advisable not to take this limit too early since it is highly non-uniform. In particular the limits volume  $\rightarrow \infty$  and frequency  $\rightarrow 0$  do not commute with each other and with the various integration processes involved [4]. Thus we shall calculate what happens if we subject a finite system to an external electric field  $\sim e^{-i\omega t}$  and then discuss the various limits.

The first model we shall study is an electron bound to the origin by harmonic forces – to the electron coordinate  $\mathbf{q}(t)$  and coupled to a vector field  $\phi(\mathbf{x}, t)$  at the origin. The field could represent phonons where we disregard the distinction between longitudinal and transversal modes and the atomistic structure of the lattice. That the particle is coupled to the field at the origin (or an average value of the field around the origin) corresponds to the dipole approximation in electrodynamics. We shall not offer any physical argument for it but our motivation is just that it renders the problem soluble. The harmonic forces are only introduced to be renormalized away, that is to say to compensate the harmonic effects of the field. Finally the particle is subjected to an arbitrary external electric field  $\mathbf{E}(t)$ . Thus the system is characterized (in appropriate units) by a Lagrangian

$$L = \frac{1}{2} \int d^3x \{ \dot{\phi}_\alpha \dot{\phi}_\alpha - \phi_{\alpha,\beta} \phi_{\alpha,\beta} \} + \frac{1}{2} (\dot{q}_\alpha \dot{q}_\alpha - \omega_0^2 q_\alpha q_\alpha) + \\ - q_\alpha \int d^3x \phi_\alpha(x) c(x) + e q_\alpha E_\alpha \quad (1)$$

where  $c$  is a suitable averaging function. We shall ask for the expectation value of  $j = e\dot{q}$  for a state specified at a time before  $E(t)$  was switched on. Since  $L$  leads to linear equations of motion the initial value problem can actually be solved and this expectation value turns out to be a linear functional of  $E$ . It leads to the standard formula for the dielectric constant of an oscillator with a frictional force. The latter is obviously due to the emission of phonons by the oscillating electrons. If the (renormalized) constant of the harmonic force is made zero one obtains a finite d.c. conductivity. One often meets the question

how time reversible equations can lead to a relation  $j = \sigma E$  between quantities which transform differently under time reversal. The answer is that by specifying the conditions at  $t \rightarrow -\infty$  rather than  $t \rightarrow \infty$  a time asymmetry is introduced. In this model the time asymmetry is directly related to the one expressed by radiation reaction which is  $\sim -\dot{q}$  if there are no phonons at the beginning and  $\sim \dot{q}$  if there are none at the end. Correspondingly  $\sigma$  is positive in the former situation and negative in the latter. Whereas the first example provides a model for the classical discussions of conductivity the second shows the typical quantum aspects of the problem. It consists of an electron interaction with scattering centers which are in some way randomly distributed [5]. It is known that for a regular arrangement of scattering centers the conductivity is  $\infty$  or 0, depending on whether the energy of the electrons is in an allowed band or not. This is a typical wave phenomenon and depends on whether the scattered waves interfere constructively or destructively. If the centers are not completely regularly arranged one may obtain a finite d.c. conductivity. In this case one cannot obtain the complete answer of the current as a functional of the electric field and we shall restrict ourselves to an expansion with respect to the external field up to linear terms. Correspondingly the question of the energy balance is not as transparent as in the first case where the Joule heat appears in the form of radiated phonons. Here the static scattering centers cannot absorb energy and one wonders where does the energy go. However the Joule heat is quadratic in the external field and does not appear in the linear approximation. This question is only answered easily for a regular arrangement in the effective mass approximation. There the external field accelerates the electron until its effective mass becomes negative then the quasi-momentum decreases again. This means that eventually Umklapp processes occur and the electron is reflected back. If the electron was initially at rest this happens only after the field has acted for a while and does not appear in the linear response which gives  $\sigma(\omega = 0) = \infty$  in this case.

## 2. THE PHONON MODEL

Our system is characterized by the Lagrangian (1) which implies the following commutation relations for equal times:



$$\begin{aligned}
[q_\alpha, \dot{q}_\beta] &= i\delta_{\alpha\beta}, & [\phi_\alpha(x), \dot{\phi}_\beta(x')] &= i\delta_{\alpha\beta}\delta(x-x') \\
[q_\alpha, q_\beta] &= [\dot{q}_\alpha, \dot{q}_\beta] = [\phi_\alpha(x), \phi_\beta(x')] = [\dot{\phi}_\alpha(x), \dot{\phi}_\beta(x')] = 0, & (2) \\
[q_\alpha, \dot{\phi}_\beta(x')] &= [\dot{q}_\alpha, \phi_\beta(x')] = [q_\alpha, \dot{\phi}_\beta(x')] = [\dot{q}_\alpha, \dot{\phi}_\beta(x')] = 0. & (3)
\end{aligned}$$

At the beginning we shall assume a finite volume with some boundary conditions which select certain values for the wave vector  $k$  in the Fourier decomposition of  $\phi(x)$ . The equations of motion originating from (1) are linear equations with constant coefficients and can be solved immediately. If we indicate the Fourier transform of the various quantities by replacing the argument  $x$  by  $k$  and  $t$  by  $\omega$  the Euler equations of (1) are

$$\begin{aligned}
(k^2 - \omega^2)\phi_\alpha(k, \omega) &= c(k)q_\alpha(\omega) & (4) \\
(\omega_0^2 - \omega^2)q_\alpha(\omega) &= \sum_k \phi_\alpha(k)c(k) + eE(\omega).
\end{aligned}$$

We shall be interested in solutions of these equations satisfying certain initial conditions. In particular we shall express all operators in terms of the field  $*$  for  $t \rightarrow -\infty$   $\phi^{\text{in}}$  and go to infinite volume so that all  $k$ -values become allowed. Then (4) becomes  $(\phi^{\text{in}}(k, \omega) \sim \delta(k^2 - \omega^2)$ ,  $\omega^\pm = \omega \pm i\epsilon)$

$$\begin{aligned}
\phi(k, \omega) &= \phi^{\text{in}}(k, \omega) + \frac{c(k)q(\omega)}{k^2 - \omega_+^2} \\
D(\omega^2)q(\omega) &\equiv (\omega_0^2 - \omega^2)q(\omega) - \int d^3k \frac{c^2(k)}{k^2 - \omega_+^2} q(\omega) & (5) \\
&= \int d^3k c(k)\phi^{\text{in}}(k) + eE(\omega).
\end{aligned}$$

The function  $D$  still depends on our choice of the cut-off function  $c$ . For  $c^2(k) = \gamma^2 M^2 / (M^2 + k^2)$ , for instance we have

$$D(z) = \omega_0^2 - z - \int \frac{d^3k c^2(k)}{k^2 - z} = \omega_0^2 - z - \frac{2\pi^2 \gamma^2 M^2}{M - i\sqrt{z}}. \quad (6)$$

If  $M \gg \omega$  we have

\* Actually one can also express everything in terms of  $\phi$ ,  $q$ , and their time derivatives at an arbitrary time and derive (5) as limiting expression. See f.i. F. Schwabl, W. Thirring, Quantum Theory of Laser Radiation, Ergebnisse der exakten Naturwissensch. **36** (1964) 219-242.

$$D_{\pm}(\omega^2) = D((\omega \pm i\varepsilon)^2) \sim \bar{\omega}^2 - \omega^2 \pm 2i\Gamma\omega \quad (7)$$

$$\bar{\omega}^2 = \omega_0^2 - 2\pi^2\gamma^2 M, \quad \Gamma = \pi^2\gamma^2$$

which shows that  $q$  obeys an oscillator equation with a friction force  $^* \sim -\dot{q}$ .

An explicit expression of the operators at any time in terms of  $\phi^{\text{in}}$  is given by

$$\phi_{\alpha}(k, t) = \int d^3k' \{ (k|\Omega_+|k')\phi_+^{\text{in}}(k', t) + (k|\Omega_-|k')\phi_-^{\text{in}}(k', t) \} + \quad (8)$$

$$+ \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{D(\omega^2)} \frac{eE(\omega)}{k^2 - \omega_+^2}$$

$$Q(t) = \int d^3k \{ f_+(k)\phi_+^{\text{in}}(k, t) + f_-(k)\phi_-^{\text{in}}(k, t) \} + \int \frac{d\omega e^{-i\omega t}}{2\pi} \frac{eE(\omega)}{D(\omega)}.$$

$\phi_{\pm}^{\text{in}}$  is the positive and negative frequency part of  $\phi^{\text{in}}$  respectively. The wave matrix  $\Omega$  and the wave function  $f$  are given by

$$(k|\Omega_{\pm}|k') = \delta(k-k') + \frac{c(k)c(k')}{k^2 - k'^2 \pm i\varepsilon} \frac{1}{D_{\pm}(k')} \quad (9)$$

$$f_{\pm}(k) = \frac{c(k)}{D_{\pm}(k)}.$$

$\Omega$  is half-sided unitary and  $f$  is normalized and orthogonal\*\* to  $\Omega$ :

$$\Omega_{\pm} \Omega_{\pm}^{\dagger} = 1, \quad \Omega_{\pm}^{\dagger} \Omega_{\pm} = 1 - f_{\mp} f_{\pm} \quad (10)$$

$$f_{\pm} \Omega_{\pm}^{\dagger} = \Omega_{\pm} f_{\mp} = 0.$$

These equations insure that the commutation relations (3) are satisfied if  $\phi^{\text{in}}$  satisfies the ones of a free field

$$[\phi_+^{\text{in}}(k, t), \phi_-^{\text{in}}(k', t)] = \frac{\delta^3(k-k')}{2k}, \quad (11)$$

$$[\phi_{\pm}^{\text{in}}(k, t), \phi_{\pm}^{\text{in}}(k', t)] = 0.$$

Thus we have a complete solution of the quantum mechanical problem. The Hilbert space is spanned by states of a definite number of

\* This is where an essential difference between this case and quantum electrodynamics in the dipole approximation appears. There this term is  $\sim \ddot{q}$ .

\*\* For a proof of this kind of relations see (6).

phonons at  $t = -\infty$ . The ground state is defined by

$$\phi_+^{\text{in}}(k, t)|_0^{\text{in}}\rangle = 0 \quad (12)$$

and the phonon states are created by applying  $\phi_-^{\text{in}}$  onto  $|_0^{\text{in}}\rangle$ . They are eigenstates of the Hamiltonian for  $E = -\infty$ . The electron is not represented by independent variables.

After these preliminaries we are in the position to deduce the conductivity directly from (8). Since  $\phi_{\pm}^{\text{in}}$  has vanishing expectation value for states with a definite number of phonons or a thermal distribution of them we get for the electron current in these cases

$$j_x(t) = e\langle \dot{q}_x(t) \rangle = \int dt' \sigma(t-t')E(t') \quad (13)$$

$$\sigma(\omega) = -\frac{ie^2\omega}{D(\omega^2)} = \sigma^*(-\omega^*).$$

If the renormalized frequency  $\bar{\omega}$  of the oscillator is zero e.g.  $\omega_0^2 = 2\pi^2\gamma^2 M$ , we obtain a finite d.c. conductivity:

$$\sigma(0) = \frac{e^2}{2\pi^2\gamma^2}. \quad (14)$$

If  $\omega \ll M$  our approximate form of  $D$  gives the elementary expression of  $\sigma$  with the polarizability of an oscillator with damping

$$\sigma(\omega) = \frac{ie^2\omega}{\omega^2 - \bar{\omega}^2 + 2i\Gamma\omega}. \quad (15)$$

It is easily seen that by integrating the equations of motion with advanced Green-functions and calculating  $\sigma$  with  $|\text{out}\rangle$  states  $\sigma(\omega)$  goes over into  $-\sigma(-\omega)$  and thus  $\sigma(0)$  changes sign.

### 3. THE IMPURITY MODEL

In this case the dynamical variables are just the electron coordinates  $\mathbf{q}$ . As potential we take (attractive) separable potentials at positions  $\mathbf{a}$  with strength  $\lambda\mathbf{a}$ .

$$V(\mathbf{x}, \mathbf{x}') = -\sum_{\mathbf{a}} \lambda_{\mathbf{a}} \rho(|\mathbf{x}-\mathbf{a}|) \rho(|\mathbf{x}'-\mathbf{a}|). \quad (16)$$

Now  $\langle j \rangle$  will be a nonlinear functional of  $E$  but we restrict ourselves

to linear terms. This is done by a perturbation treatment of  $H'$  where

$$\begin{aligned} H &= H_0 + H' \\ H_0 &= \frac{1}{2}p^2 + V \\ H' &= e\mathbf{q} \cdot \mathbf{E}(t) \text{ or } -e\mathbf{p} \cdot \mathbf{A}(t) \end{aligned} \quad (17)$$

depending on whether the electric field is represented by a scalar or vector potential.  $\sigma$  can then be calculated in a fashion familiar from the Heisenberg-Kramers dispersion theory\*

$$(H_0|\gamma\rangle = E_\gamma|\gamma\rangle; \langle\gamma|j(\omega)|\gamma\rangle = \sigma_\gamma(\omega)E(\omega); \text{Im } \omega > 0) \quad (18)$$

$$\sigma_\gamma(\omega) = \frac{e^2}{3} \langle\gamma|p_\alpha \frac{1}{H_0 - E_\gamma - \omega} q_\alpha + q_\alpha \frac{1}{H_0 - E_\gamma + \omega} p_\alpha|\gamma\rangle$$

or

$$\sigma_\gamma(\omega) = \frac{ie^2}{3\omega} \langle\gamma|3 - p_\alpha \frac{1}{H_0 - E_\gamma - \omega} p_\alpha - p_\alpha \frac{1}{H_0 - E_\gamma + \omega} p_\alpha|\gamma\rangle.$$

The two forms of  $\sigma$  originate from the two forms of  $H'$  and are equivalent of the following sum rule holds

$$\delta_{\alpha\beta} = \langle\gamma|p_\alpha \frac{1}{H_0 - E_\gamma} p_\beta + p_\beta \frac{1}{H_0 - E_\gamma} p_\alpha|\gamma\rangle \quad (19)$$

which follows under certain conditions from the canonical commutation rules

$$[q_\alpha, p_\beta] = i\delta_{\alpha\beta}. \quad (20)$$

For periodic boundary condition (20) and (19) fail to hold since  $q$  is not an operator in the Hilbert space of periodic functions. In this case the second form of  $H'$  and  $\sigma$  has to be used\*\*.

From (18) it appears that  $\sigma(\omega)$  can be continued analytically into the complex  $\omega$  plane with poles on the real axis. Furthermore we see from the second form of (18)

$$\sigma(\omega) = \sigma^*(-\omega^*) = -\sigma(-\omega) \quad (21)$$

However in the limit volume  $\rightarrow \infty$  the eigenvalue spectrum of  $H_0$  may become continuous and  $\sigma$  will acquire a cut along the real axis. Then

\* Actually, since for our nonlocal potential  $[q, V] \neq 0$  there are other contributions to the current. However, they go to zero in the limit of small range of  $\rho$  which we will consider.

\*\* In our previous model (19) holds and both forms of  $\sigma$  give (13).



for  $\omega$  in the lower half plane  $\sigma$  given by (18) may not be the analytic continuation of  $\sigma$  from the upper half plane. In this case  $\sigma(\omega) = -\sigma(-\omega)$  is no longer true for the analytically continued function and we may have  $\sigma(0) \neq 0$ . If (19) holds we find

$$\sigma(0) = -\frac{2}{3}\pi e^2 \langle \gamma | p_\alpha \delta'(H_0 - E_\gamma) p_\alpha | \gamma \rangle. \quad (22)$$

The evaluation of (18) can be carried out with the aid of the Greens function

$$G_E(x, x') = \langle x | \frac{1}{H_0 - E} | x' \rangle. \quad (23)$$

On expanding this into powers of  $V$  we arrive at a geometrical series in  $V$  provided  $\rho$  is a step function in momentum space.

$$\rho^2(p) = \rho(p) \quad (24)$$

$$\rho(p) = \begin{cases} 1 & \text{for } p < A \\ 0 & \text{for } p > A. \end{cases}$$

In this case we find

$$G(x, x') = G_0(x - x') + \sum_{a, a'} \bar{G}_0(x - a) M_{aa'} \bar{G}_0(a - x') \quad (25)$$

with

$$G_0(x) = \frac{e^{-\kappa x}}{2\pi x}, \quad \kappa = \sqrt{-2E} \quad (26)$$

$$\bar{G}_0(x) = \int dx' \rho(x') \frac{e^{-\kappa|x-x'|}}{2\pi|x-x'|}.$$

The matrix  $M$  is given by

$$M = (\lambda^{-1} - R)^{-1} \quad (27)$$

$$\lambda_{aa'} = \delta_{aa'} \lambda_a, \quad R_{aa'} = \bar{G}_0(a - a')$$

and it is this matrix inversion where the difficulty is hidden. If all  $\lambda$  are equal and the  $a$ 's form a cubic lattice with lattice constant unity and  $N a$ 's in a row we find in the tight binding limit \*  $e^{-\kappa} \ll 1$

\* In this case  $G$  looked at from a birds eye view is  $G_0$  for a particle with an effective mass  $\frac{1}{2}e^{-\kappa}$ .

$$M_{a,a'} = \lambda \sum_{s_i = \pi/(N+1)}^{N\pi/(N+1)} \left( \frac{2}{N+1} \right)^3 \left[ 1 - \lambda c(\kappa) - \frac{\lambda e^{-\kappa}}{\pi} (\cos s_1 + \cos s_2 + \right. \\ \left. + \cos s_3) \right]^{-1} \cdot \prod_{\alpha=1}^3 \sin a_\alpha s_\alpha \cdot \sin a'_\alpha s_\alpha; \\ c(\kappa) = \pi^{-2} [A - \kappa \operatorname{arctg} A/\kappa].$$

The next quantity we need for calculating  $\sigma$  is (for  $A \rightarrow \infty$ )

$$f(\omega) = \frac{1}{3} \sum_{\gamma} \delta(E - E_{\gamma}) \langle \gamma | p_{\alpha} (H_0 - E_{\gamma} + \omega)^{-1} p_{\alpha} | \gamma \rangle \\ = \frac{1}{3\pi} \int \frac{d^3 p d^3 p'}{(2\pi)^6} (\mathbf{p} \cdot \mathbf{p}') \bar{G}_E(p, p') G_{E-\omega}(p', p) \\ = \frac{1}{3\pi} \int \frac{d^3 p d^3 p'}{(2\pi)^6} (\mathbf{p} \cdot \mathbf{p}') \sum_{a,b,a'b'} \left\{ 2 \frac{(2\pi)^3 \delta(p-p')}{p^2 + \bar{\kappa}^2} + \right. \\ \left. + \frac{4M_{a'a}(\bar{\kappa}) e^{i(p'a' - pa)}}{(p^2 + \bar{\kappa}^2)(p'^2 + \bar{\kappa}^2)} \right\} \cdot \frac{4\bar{M}_{bb'}(\kappa) e^{i(pb - p'b')}}{(p^2 + \kappa^2)(p'^2 + \kappa^2)} \quad (29)$$

with

$$\kappa^2 = \kappa^2 - 2\omega, \\ \bar{G}_E = \frac{1}{2i} (G_{E+i\eta} - G_{E-i\eta}), \quad \bar{M} = \frac{1}{2i} (M(E+i\eta) - M(E-i\eta)) \quad (30)$$

in the limit  $\eta \rightarrow 0$ .

The sum rule (19) implies

$$f(0) = \frac{1}{2} \sum_{\gamma} \delta(E - E_{\gamma}) \quad (31)$$

in which case we have

$$e^{-2} \sum_{\gamma} \sigma_{\gamma} \delta(E - E_{\gamma}) = \frac{1}{i\omega} (f(+\omega) + f(-\omega) - 2f(0)). \quad (32)$$

Since the general evaluation of  $M$  is impossible we specialize now to one typical situation. We take the  $a$ 's again to form a cubic lattice but let the  $\lambda_a$ 's be distributed in some manner. Thus the bound state of a single separable potential will be spread out into an impurity band and we shall work out the conductivity in this band. For this purpose we decompose  $M^{-1}$  into a diagonally part  $D_{aa'} = \delta_{aa'}((1/\lambda_a) - C(\kappa))$

and a nondiagonal part  $K$  and expand in powers of  $K$

$$M = (D - K)^{-1} = D^{-1} + D^{-1}KD^{-1} + D^{-1}KD^{-1}KD^{-1} + \dots \quad (33)$$

In this manner we obtain

$$M_{aa'} = \frac{\delta_{aa'}\lambda_a}{1 - \lambda_a c} + \frac{\lambda_a \lambda_{a'}}{2\pi(1 - \lambda_a c)(-\lambda_{a'} c)} \{e^{-\kappa} \sum_n \delta_{a+n, a'} +$$

$$+ e^{-\kappa\sqrt{2}} \sum_{n'} \delta_{a+n', a'} + \dots\} + \frac{e^{-2\kappa}}{4\pi^2} \delta_{aa'} \left(\frac{\lambda_a}{1 - \lambda_a c}\right)^2 \sum_n \frac{\lambda_{a+n}}{1 - \lambda_{a+n} c} + \dots, \quad (34)$$

$$\frac{1}{\pi} \bar{M}_{aa'} = \delta_{aa'} \lambda_a \delta(1 - \lambda_a c) + \frac{e^{-\kappa}}{2\pi} \sum_n \delta_{a'+n, a} \frac{\lambda_a \lambda_{a+n}}{c(\lambda_a - \lambda_{a+n})} \cdot$$

$$\cdot (\delta(1 - \lambda_a c) - \delta(1 - \lambda_{a+n} c)) \dots + \delta_{aa'} \frac{e^{-2\kappa}}{4\pi^2} \sum_n \lambda_a^2 \lambda_{a+n} \cdot$$

$$\cdot \left\{ \frac{1}{c^2(\lambda_{a+n} - \lambda_a)^2} (\delta(1 - \lambda_{a+n} c) - \delta(1 - \lambda_a c)) - \frac{\delta'(1 - \lambda_a c)}{c(\lambda_a - \lambda_{a+n})} \right\} + \dots$$

Here  $\sum_n$  always means sum over the next neighbours,  $\sum_{n'}$  sum over the second neighbours, etc. Our expansion is essentially an expansion in  $e^{-\kappa}$ , more exactly it makes sense in a tight binding situation if the  $\lambda_a$  are distributed such that

$$e^{-\kappa|n|} \ll \frac{|\lambda_a - \lambda_{a+n}|}{\lambda_a^2}. \quad (35)$$

One readily verifies up to order  $e^{-2\kappa}$  that (31) is indeed satisfied and thus (32) can be used to calculate  $\sigma$ . If  $f(\omega)$  were analytic in  $\omega$  near the origin it is clear that for small  $\omega$  we would get  $\sigma \sim \omega$ , e.g. there would be no d.c. conductivity but only a polarizability. From (29) we see that  $f$  depends on  $\omega$  via  $\bar{\kappa}$  which appears in the denominator and in  $M$ . For small  $\omega$  the combination  $(p^2 + \kappa^2 - 2\omega)^{-1}$  is perfectly analytic and it must be through  $M(\bar{\kappa})$  that we get a  $\sigma(0)$ . Thus for a small  $\omega$  the leading term of  $f$  will be\*

$$f_{II} = \frac{16}{3\pi} \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{(p \cdot p') e^{i[p'(a' - b')p - (a - b)]}}{(p^2 + \kappa^2)^2 (p'^2 + \kappa^2)^2} - M_{a'a}(\bar{\kappa}) \bar{M}_{bb'}(\kappa) \quad (36)$$

\* The other part gives mainly the polarizability of the electron in the separable potentials.

where in the denominator we dropped the  $\omega$ . Because of

$$\int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{(p \cdot p') e^{i(p x - p' x')}}{(p^2 + \kappa^2)^2 (p'^2 + \kappa^2)^2} = \frac{(x \cdot x') e^{-\kappa(|x| + |x'|)}}{64\pi^2 |x| \cdot |x'|} \quad (37)$$

it turns out that the contribution of lowest order in  $e^{-\kappa}$  appears if  $a = a'$  and  $b = b'$  are neighbours in which case we obtain

$$f_{II}(\omega) = \frac{e^{-2\kappa}}{12\pi^2} \sum_{n,a} \delta(1 - \lambda_a c(\kappa)) \frac{\lambda_a \lambda_{a+n}}{1 - \lambda_{a+n} c(\bar{\kappa})}. \quad (38)$$

To recover  $\sigma$  we have to write the  $\delta$ -function in the form  $\delta(E - E_\gamma) = (\delta/\kappa)(\kappa - \kappa_\gamma)$ . This is done if we remember that for  $A \gg \kappa$

$$c(\kappa) = \frac{-\kappa + \kappa(\lambda)}{2\pi} + \lambda^{-1}, \quad c(\bar{\kappa}) = c(\kappa) + \frac{\omega}{2\pi\kappa} \quad (39)$$

$\kappa(\lambda)$  being the eigenvalue of the separable potentials with strength  $\lambda$ . This gives us finally

$$f_{II}(\omega) = \frac{1}{6\pi} \sum_{a,n} \frac{1}{\kappa} \delta(\kappa - \kappa(\lambda_a)) \frac{e^{-2\kappa} \lambda_a \lambda_{a+n}}{\lambda_a - \lambda_{a+n} - (\lambda_a^2 \omega / 2\pi\kappa)}. \quad (40)$$

Thus if we want  $\sigma$  in a certain energy region  $\Delta E$ , e.g.  $\sum_{E_\gamma \in \Delta E} \sigma_\gamma$  we have to sum in (30) over those  $a$  where  $\lambda_a$  produces a state in  $\Delta E$  and of all neighbours of these  $a$ . We shall assume that the  $\lambda$ 's are concentrated around an average value  $\lambda_0$  according to a distribution

$$P(\lambda) = \frac{\gamma/\pi}{(\lambda - \lambda_0)^2 + \gamma^2}. \quad (41)$$

In a macroscopic piece of our system ( $N \rightarrow \infty$ ) the sum in (40) will consist of so many terms that we can replace  $\sum_n$  by

$$6 \int_{-\infty}^{\infty} d\lambda_{a+n} P(\lambda_{a+n}).$$

There is just one complication owing to condition (35) outside of which our expansion is useless. It requires  $|\lambda_a - \lambda_{a+n}| \geq \delta$  with  $\delta \sim e^{-\kappa}$ . Thus we cannot take  $\lambda_{a+n}$  independent of  $\lambda_a$  but have to take



$$\left( \int_{-\infty}^{\lambda_a - \delta} + \int_{\lambda_a + \delta}^{\infty} \right) d\lambda_{a+n}.$$

Instead of  $\int_{-\infty}^{\infty} d\lambda_{a+n}$ . With the distribution (41) the integrals in (40) are elementary and we identify  $\sigma$  as the coefficient of the  $\delta$ -function on inserting (40) into (32)

$$\begin{aligned} e^{-2} \text{Re } \sigma(\omega) &= \frac{\lambda_a^4 e^{-2\kappa\gamma} 2\gamma(\lambda_a - \lambda_0)}{\pi^2 [(\lambda_a - \lambda_0)^2 + \gamma^2]^2} \quad \text{for } \omega \frac{\lambda_a^2}{2\pi\kappa_r} > \delta \\ &= 0 \quad \text{for } < \delta. \end{aligned} \quad (42)$$

Here again the value of  $\lambda_a$  is such that  $\kappa(\lambda_a)$  corresponds to the energy of the state for which we calculate  $\sigma$ . Because of  $\sigma(\omega) = \sigma^*(-\omega^*)$  for  $\omega = 0$   $\sigma$  will be real,  $\text{Im } \sigma$  being  $\sim \omega$ . Owing to our limitation (35) we got  $\sigma(0) = 0$ . However, in the tight binding limit  $\delta$  can be made arbitrarily small and thus our result is \*

$$\lim_{\omega \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{N \rightarrow \infty} \sigma = e^2 \frac{\lambda_a^4 e^{-2\kappa} 2\gamma(\lambda_a - \lambda_0)}{\pi^2 [(\lambda_a - \lambda_0)^2 + \gamma^2]^2}. \quad (43)$$

The significance of the various factors in (43) is the following. Since we put the lattice constant = 1 or  $\kappa = (\text{Distance between atoms/atomic radius})$  the factor  $e^{-\kappa}$  is related to the fact that the electron has to tunnel from one atom to the next. More in detail we can argue as follows.

We expect  $\sigma$  to be

$$\sigma = \frac{e^2}{m^* v |a|^2}$$

with  $m^*$  = effective mass,  $v$  = velocity,  $|a|^2$  = scattering cross-section. For the scattering of a particle of mass  $m^*$  on a separable potential with strength  $\lambda_a - \lambda_0$  we have  $a \sim (\lambda_a - \lambda_0) m^*$ . Since  $m^* \sim e^\kappa$  we get  $\sigma \sim e^{-2\kappa}$  provided we keep  $m^* v = k$  = quasimomentum constant. The denominator in (43) shows that the conductivity is better near the center of the band since there it is more probable that neighbouring  $\lambda$ 's have close values. Remarkable is the factor  $\lambda_a - \lambda_0$  in the numerator which shows that  $\sigma$  changes sign on the top side of the band. Thus we get hole conduction in spite of the diffuse nature of the

\* This can also directly be found with (22).

band. One might wonder whether one would not obtain (43) by starting from the Bloch functions as given by (28) if all  $\lambda_a$  are equal. However the width of the Bloch band is  $\sim \delta$  and our condition (35) just tells us that the width  $\gamma$  due to the irregularity of the  $\lambda$ 's has to be much larger than  $\delta$ . Nevertheless, as we have seen the qualitative features of the band picture are still present.

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# THE MORAL ASPECT OF QUANTUM MECHANICS

J. S. BELL

*CERN, Geneva*

and

M. NAUENBERG

*Stanford University*

*(Received June 3, 1965)*

The notion of morality appears to have been introduced into quantum theory by Wigner, as reported by Goldberger and Watson [1]. The question at issue is the famous "reduction of the wave packet". There are, ultimately, no mechanical arguments for this process, and the arguments that are actually used may well be called moral. This is a popular account of the subject. Very practical people not interested in logical questions should not read it. It is a pleasure for us to dedicate the paper to Professor Weisskopf, for whom intense interest in the latest developments of detail has not dulled concern with fundamentals.

Suppose that some quantity  $F$  is measured on a quantum mechanical system, and a result  $f$  obtained. Assume that immediate repetition of the measurement must give the same result. Then, after the first measurement, the system must be in an eigenstate of  $F$  with eigenvalue  $f$ . In general, the measurement will be "incomplete", i.e., there will be more than one eigenstate with the observed eigenvalue, so that the latter does not suffice to specify completely the state resulting from the measurement. Let the relevant set of eigenstates be denoted by  $\phi_{fg}$ . The extra index  $g$  may be regarded as the eigenvalue of a second observable  $G$  that commutes with  $F$  and so can be measured at the same time. Given that  $f$  is observed for  $F$ , the relative probabilities of observing various  $g$  in a simultaneous measurement of  $G$  are given by the squares of the moduli of the inner products

$$(\phi_{fg}, \psi)$$

where  $\psi$  is the initial state of the system. Let us now make the plausible assumption that these relative probabilities would be the same if  $G$

were measured not simultaneously with  $F$  but immediately afterwards. Then we know something more about the state resulting from the measurement of  $F$ . One state with the desired properties is clearly

$$N \sum_g \phi_{fg}(\phi_{fg}, \psi)$$

where  $N$  is a normalization factor. It is readily shown that this is the only state [2] for which the probability of obtaining a given value for *any* quantity commuting with  $F$  is the same whether the measurement is made at the same time or immediately after. Thus, we arrive at the general formulation for the "reduction of the wave packet" following measurement [3]: expand the initial state in eigenstates of the observed quantity, strike out the contributions from eigenstates which do not have the observed eigenvalue, and renormalize the remainder. This preserves the original phase and intensity relations between the relevant eigenstates. It therefore does the minimum damage to the original state consistent with the requirement that an immediate repetition of the measurement gives the same result. All this is very ethical, and we will refer to the particular reduction just defined as "the moral process".

Now morality is not universally observed, and it is easy to think of measuring processes for which the above account would be quite inappropriate. Suppose for example the momentum of a neutron is measured by observing a recoil proton. The momentum of the neutron is altered in the process, and in a head on collision actually reduced to zero. The subsequent state of the neutron is by no means a combination (the spin here provides the degeneracy) of states with the observed momentum. How then is one to know whether a given measurement is moral [4] or not? Clearly, one must investigate the physics of the process. Instead of tracing through a realistic example we will follow von Neumann [3] here in considering a simple model.

Suppose the system I to be observed has co-ordinates  $R$ . Suppose that the measuring instrument, II, has a single relevant co-ordinate  $Q$ —a pointer position. Suppose that the measurement is effected by switching on instantaneously an interaction between I and II

$$\delta(t)F\left(R, \frac{1}{i}\frac{\partial}{\partial R}\right)\frac{1}{i}\frac{\partial}{\partial Q}$$



where  $t$  is time. The simplification here, where the system of interest acts directly on a pointer reading without intervention of circuitry, is gross. If I is in the state  $\psi(R)$  before the measurement, and the pointer reading is zero, the initial state of I+II is

$$\psi(R)\delta(Q).$$

The state of I+II immediately after  $t = 0$  can be obtained by solving the Schrödinger equation. In this only the interaction term in the Hamiltonian is significant, because of its impulsive character. The resulting state is [5]

$$\sum_{f,g} \phi_{fg}(R)(\phi_{fg}, \psi)\delta(Q-f)$$

where  $f$  is an eigenvalue of  $F$ ,  $\phi_{fg}$  a corresponding eigenfunction, and  $g$  any extra index needed to enumerate these eigenfunctions. If now an observer reads the pointer on the instrument, and finds a particular value  $f$ , *and if this measurement of the pointer reading is moral*, then the state reduces to

$$N \sum_g \phi_{fg}(R)(\phi_{fg}, \psi)\delta(Q-f).$$

The part referring to system I alone,

$$N \sum_g \phi_{fg}(R)(\phi_{fg}, \psi)$$

is precisely the result of applying the moral process to I directly, after the measurement of the quantity  $F$ . So we have here a dynamical model of a moral measurement of  $F$ . This depends on the detailed nature of the interaction between the system and the measuring instrument. It would have been equally easy to choose an interaction for which a moral measurement of the pointer reading would imply an immoral measurement of  $F$ .

Thus, if the morality of measurements of macroscopic pointer readings is granted, there is no real ambiguity in practice in applying quantum mechanics. One must simply understand well enough the structure of the systems involved, including the instruments, and work out the consequences. This situation is not peculiar to quantum mechanics. Moreover, we are readily disposed to accept the moral character of observing macroscopic pointers, for we feel convinced from common

experience that they are not much changed in state by being looked at, and the moral process is in an obvious sense minimal. Thus, the basis of practical quantum mechanics seems secure. This is just as well, in view of its magnificent success, and of the fact that there is no real competitor in sight. However, it must not be supposed that the action on the wave function of even such a macroscopic observation is of a trivial nature, and least of all that it is a mere subjective adjustment of the representative ensemble to allow for increased knowledge. To make this elementary point suppose that the measuring interaction in the above model is again switched on at times  $\tau$  and  $2\tau$ :

$$\delta(t-\tau)F \frac{1}{i} \frac{\partial}{\partial Q}, \quad \delta(t-2\tau)F \frac{1}{i} \frac{\partial}{\partial Q}.$$

During the period  $\tau$  suppose that each eigenstate  $\phi_f$  (the possible extra index  $g$  is not essential here) evolves into a combination

$$\phi_f \rightarrow \sum_{f'} \phi_{f'} \alpha_{f', f}.$$

For the instrument II suppose for simplicity that  $Q$  is a constant of the motion between interactions. Then solution of the Schrödinger equation for I+II gives from the initial state (just before  $t = 0$ )

$$\psi \delta(Q)$$

the final state

$$\sum_{f, f', f''} \phi_{f''} \alpha_{f'', f'} \alpha_{f', f} (\phi_f, \psi) \delta(Q - f - f' - f'')$$

just after  $t = 2\tau$ . The probabilities of then observing various particular possible values  $Q$  for the pointer position are given by

$$\sum_{f''} \left| \sum_f \alpha_{f'', Q-f-f''} \alpha_{Q-f-f'', f} (\phi_f, \psi) \right|^2.$$

Now this assumes that the intermediate evolution of I+II is governed entirely by the Schrödinger equation, and therefore *that the pointer position is not looked at until after the final interaction*. If the pointer position is observed just after *each* interaction then the moral process comes into play just after  $t = 0$  and  $t = \tau$ . If all possible results of these intermediate observations are averaged over the net result is simply to eliminate from the last expression interference between

different values of  $f$  and  $f'$ ; it becomes

$$\sum_{f''} \sum_f |\alpha_{f'', Q-f-f''} \alpha_{Q-f-f'', f}(\phi_f, \psi)|^2.$$

Thus observation, even when all possible results are averaged over, is a dynamical interference with the system which may alter the statistics of subsequent measurements.

Now although we would not wish to cast doubt on the *practical* adequacy of macroscopic morality, it is clear that if we leave it unanalyzed the theory can at best be described as a phenomenological makeshift. The fact already stressed that observation implies a dynamical interference, together with the belief that instruments after all are no more than large assemblies of atoms, and that they interact with the rest of the world largely through the well-known electromagnetic interaction, seems to make this a distinctly uncomfortable level at which to replace analysis by axioms. The only possibility of further analysis offered by quantum mechanics is to incorporate still more of the world into the quantum mechanical system, I + II + III + etc. Especially from the theorist's point of view such a development is very pertinent. For him the experiment may be said to start with the printed proposal and to end with the issue of the report. For him the laboratory, the experimenter, the administration, and the editorial staff of the Physical Review, are all just part of the instrumentation. The incorporation of (presumably) conscious experimenters and editors into the equipment raises a very intriguing question. For they know the results before the theorist reads the report, and the question is whether their knowledge is incompatible with the sort of interference phenomena discussed above. If the interference is destroyed, then the Schrödinger equation is incorrect for systems containing consciousness. If the interference is not destroyed the quantum mechanical description is revealed as not wrong but certainly incomplete [8]. We have something analogous to a two-slit interference experiment where the "*particle*" in any particular instance has gone through only one of the slits (and knows it!) and yet there are interference terms depending on the *wave* having gone through both slits. Thus we have *both waves and particle trajectories*, as in the de Broglie-Bohm "pilot wave" or "hidden parameter" interpretations of quantum mechanics [7]. Unfortunately it seems hopelessly impossible to test this

question in practice; it is hard enough to realize interference phenomena involving simple things like electrons, photons, or  $\alpha$  particles. Experimenters (and even inanimate instruments) radiate heat, for example, and this coupling to their surroundings suppresses interference just as effectively as the theorist reading the *Physical Review*. Nevertheless, the question of principle is there. Now, even if we had settled the status of the experimenter, we are not at the end of the road. For the reading of the *Physical Review* is hardly a more elementary act than the reading of pointers or computer output; this act also seems to require analysis rather than axiomatics, and so we want the theorist also in the Schrödinger equation. He also radiates heat, and so on, and we want finally the whole universe in the quantum mechanical system. At this point we are finally lost. It is easy to imagine a state vector for the whole universe, quietly pursuing its linear evolution through all of time and containing somehow all possible worlds. But the usual interpretive axioms of quantum mechanics come into play only when the system interacts with something else, is "observed". For the universe there *is* nothing else, and quantum mechanics in its traditional form has simply nothing to say. It gives no way of, indeed no meaning in, picking out from the wave of possibility the single unique thread of history.

These considerations, in our opinion, lead inescapably to the conclusion that quantum mechanics is, at the best, incomplete [8]. We look forward to a new theory which can refer meaningfully to events in a given system without requiring "observation" by another system. The critical test cases requiring this conclusion are systems containing consciousness and the universe as a whole. Actually, the writers share with most physicists a degree of embarrassment at consciousness being dragged into physics, and share the usual feeling that to consider the universe as a whole is at least immodest, if not blasphemous. However, these are only logical test cases. It seems likely to us that physics will have again adopted a more objective description of nature long before it begins to understand consciousness, and the universe as a whole may well play no central role in this development. It remains a logical possibility that it is the act of consciousness which is ultimately responsible for the reduction of the wave packet [9]. It is also possible that something like the quantum mechanical state



function continue to play a role, supplemented by variables describing the actual as distinct from the possible course of events ("hidden variables") although this approach seems to face severe difficulties in describing separated systems in a sensible way [7]. What is much more likely is that the new way of seeing things will involve an imaginative leap that will astonish us. In any case it seems that the quantum mechanical description will be superseded. In this it is like all theories made by man. But to an unusual extent its ultimate fate is apparent in its internal structure. It carries in itself the seeds of its own destruction.

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$$\chi = \phi_{fg} \delta(Q - \alpha(t)f)$$

satisfies

$$\frac{\partial \chi}{\partial t} = - \frac{d\alpha}{dt} f \frac{\partial \chi}{\partial Q} = -i \frac{d\alpha}{dt} F \frac{1}{i} \frac{\partial \chi}{\partial Q}.$$

So we need  $(d\alpha/dt) = \delta(t)$ , or that  $\alpha$  increases from zero to one during the interaction. Given in the text is the combination such solutions which corresponds to the prescribed initial state.

- 6) It is taken for granted here that conscious experience is of, or is, a unique sequence of events, and cannot be completely described by a quantum mechanical state containing somehow all possible sequences. Occasionally people challenge this view. The writers therefore concede that there may be *some* people

whose states of mind are best described by coherent or incoherent quantum mechanical superpositions.

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# ENERGIES AND HAMILTONIANS IN MAGNETIC FIELDS

H. B. G. CASIMIR

*Eindhoven*

(Received June 4, 1965)

1. The following simple paradox is well known in electrostatics: the energy of an electrical condensor is

$$U = \frac{1}{2}CV^2$$

where  $C$ , the capacitance is given by

$$C = \frac{A}{4\pi D}$$

or, if one prefers

$$C = \frac{\epsilon_0 A}{D}.$$

Therefore

$$\frac{\partial U}{\partial D} < 0$$

and the plates should repel one another.

The explanation is of course that a condensor at constant voltage is not a closed system; a condensor at constant charge is. We have

$$U = \frac{1}{2C}Q^2$$

and

$$\left(\frac{\partial U}{\partial D}\right)_Q = -\left(\frac{\partial U}{\partial D}\right)_V.$$

The difference

$$\delta D \left\{ \left(\frac{\partial U}{\partial D}\right)_V - \left(\frac{\partial U}{\partial D}\right)_Q \right\}$$

is supplied by the battery that keeps  $V$  constant.

More generally for a system of conductors

$$U = \frac{1}{2} \sum C_{nm} V_n V_m$$

and the charge on the  $n^{\text{th}}$  conductor is

$$Q_n = \sum C_{nm} V_m.$$

Then

$$(\delta U)_V = \frac{1}{2} \sum \delta C_{nm} V_n V_m$$

and

$$(\delta U)_Q = \frac{1}{2} \sum \delta C_{nm} V_n V_m + \sum C_{nm} V_n \delta V_m.$$

But

$$\delta Q_n = \sum \delta C_{nm} V_m + \sum C_{nm} \delta V_m = 0$$

hence

$$(\delta U)_Q = -(\delta U)_V.$$

2. For a system of *currents* in closed linear conductors we have

$$U = \frac{1}{2} \sum L_{nm} i_n i_m.$$

In this case

$$\Phi_n = \sum L_{nm} i_m$$

is the magnetic flux through the  $n^{\text{th}}$  ring. We have now

$$(\delta U)_i = -(\delta U)_\Phi.$$

For superconducting rings we have  $\delta \Phi_n = 0$ . Therefore  $(\delta U)_\Phi$  is the correct expression for a closed system and should be used to calculate forces. The difference  $(\delta U)_i - (\delta U)_\Phi$  is supplied by current sources that maintain a constant value of  $i$ .

3. Closely related and more liable to give rise to confusion is the problem of magnetic work on a body in a magnetic field.

First, let us assume that a field is produced by a superconducting coil and that this coil has a large self inductance so that  $Li$  is large compared with whatever flux may originate from the magnetic body considered. The magnetic field may be changed by moving the coil (and in this arrangement that is the only way in which magnetic work can be done). Instead of moving the field source we may also move the body. If the external field (i.e. the field that would be there in the absence of the body) does not vary appreciably over the region



of space to be occupied by this body, then the work required to move a dipole  $\mathbf{m}$  from a region with field  $\mathbf{H}_0$  to a region with field  $\mathbf{H}_0 + \delta\mathbf{H}_0$  is  $-\mathbf{m} \cdot \delta\mathbf{H}_0$  and if we start from a situation in which the body is entirely outside the field the change in energy when it is brought into the field  $\mathbf{H}$  is given by

$$\Delta U = U_1 - U_0 = - \int_0^H \mathbf{m} \cdot d\mathbf{H}_0 + Q$$

where  $Q$  is the heat supplied during magnetization. But we may also imagine that coil and body are stationary and that we slowly increase the current from 0 to a value  $i$ . This would require an energy  $\frac{1}{2}Li^2$  for an empty coil. If a magnetic body is present more energy is required: the change of  $\mathbf{m}$  induces an additional electric field against the rising current. It is easily shown that for a change  $\delta\mathbf{m}$  this extra energy is  $\mathbf{H}_0 \cdot \delta\mathbf{m}$  and for the total energy difference we have

$$\Delta W = W_1 - W_0 = \int_0^H \mathbf{H}_0 \cdot d\mathbf{m} + Q.$$

$\Delta W$  measures the energy difference between empty coil and coil plus magnetic body, compared at the same current;  $\Delta U$  measures this difference comparing empty coil and coil plus body at the same flux.

To show that there is no contradiction between these expressions consider the case of constant flux and assume that in the final state the dipole  $\mathbf{m}$  sends a flux  $\Phi_m$  through the coil. Then in order to maintain the original flux  $Li$  the current has to be changed by an amount  $\delta i$  given by

$$L\delta i = -\Phi_m.$$

Now it is easily shown that

$$\mathbf{H}_0 \cdot \mathbf{m} = i\Phi_m$$

therefore

$$iL\delta i = -\mathbf{H}_0 \cdot \mathbf{m}.$$

But  $-Li\delta i$  is exactly the energy difference that is required to bring the current in the coil to its original value. And as a matter of fact

$$W_1 - U_1 = \int_0^H \mathbf{H}_0 \cdot d\mathbf{m} + \int_0^H \mathbf{m} \cdot d\mathbf{H}_0 = \int_0^H d(\mathbf{H}_0 \cdot \mathbf{m}) = \mathbf{H}_0 \cdot \mathbf{m}.$$

4. To calculate thermodynamic equilibrium in a constant field we have to use  $U$  rather than  $W$ . Let us consider an example in which there is no heat involved. In a cylindrical coil with cross section of area  $A$  we place a superconducting rod of cross section  $a \ll A$ . Then the energy per unit length of the empty coil is

$$\frac{1}{8\pi} H^2 \cdot A$$

and the energy of coil plus rod at the same current

$$\frac{1}{8\pi} H^2 \cdot (A - a).$$

Therefore

$$\Delta W = - \frac{1}{8\pi} H^2 \cdot a.$$

But at constant flux

$$(H + \delta H)(A - a) = HA$$

and

$$\Delta U = \{(H + \delta H)^2(A - a) - H^2 \cdot A\} / 8\pi \sim \frac{1}{8\pi} H^2 \cdot a.$$

The relevant expression  $U$  increases when the superconducting rod is inserted. As far as magnetic energy is concerned the situation with Meisznereffect is disadvantageous – as it should be. Errors based on confusion of  $\Delta W$  and  $\Delta U$  – thinly disguised as variational principles – can be found in several older papers on superconductivity.

5. The energy derived from the usual Hamiltonian for a system in an external magnetic field leads to  $U$  and not to  $W$ . Therefore a first order calculation of magnetic interaction energies using zero order approximations for the currents gives a result of correct absolute value but incorrect sign.

Consider a system with energy

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M \dot{X}^2 + \alpha \dot{x} \dot{X}$$

where  $\alpha$  may be a function of  $x$  and  $X$ .

Then

$$\begin{aligned} p &\equiv p_x = m\dot{x} + \alpha\dot{X} \\ P &\equiv p_X = M\dot{X} + \alpha\dot{x} \end{aligned}$$

and the Hamiltonian is

$$\mathcal{H} = \frac{1}{1 - \alpha^2/mM} \left[ \frac{1}{2m} p^2 + \frac{1}{2M} P^2 - \alpha pP/mM \right].$$

If  $\alpha$  is small the influence of the coupling terms on energy levels will be

$$\delta E = -(\alpha pP/mM)_{av} \sim -(\alpha \dot{X} \dot{X})_{av}.$$

We can also write

$$\mathcal{H} = \frac{1}{1 - \alpha^2/mM} \left[ \frac{1}{2m} (p - \alpha P/M)^2 \right] + \frac{1}{2M} P^2.$$

If  $M$  tends to infinity and  $X$  remains finite then  $P/M$  tends toward  $\dot{X}$ . The analogy with the usual Hamiltonian for particles in a field should be obvious.

6. Magnetic interaction energy can be expressed in several ways. The basic expression is

$$W_m = \frac{1}{4\pi} \iiint \mathbf{H}_1 \cdot \mathbf{H}_2 d\tau.$$

Write

$$\mathbf{H}_2 = \text{curl } \mathbf{A}_2$$

then

$$W_m = \frac{1}{4\pi} \int \mathbf{H}_1 \cdot \text{curl } \mathbf{A}_2 d\tau = \frac{1}{4\pi} \int \mathbf{A}_2 \cdot \text{curl } \mathbf{H}_1 d\tau = \frac{1}{c} \int \mathbf{i}_1 \mathbf{A}_2 d\tau.$$

Similarly

$$W_m = \frac{1}{c} \int \mathbf{A}_1 \mathbf{i}_2 d\tau.$$

Since

$$\mathbf{A}_2 = \frac{1}{c} \int \frac{i(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} d\tau$$

we can also write

$$W_m = \frac{1}{c^2} \iint \frac{i_1(\mathbf{r}) i_2(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau d\tau'.$$

In many cases it is useful to write

$$\mathbf{i}_2 = c \operatorname{curl} \mathbf{M}_2.$$

This defines a magnetic moment density  $\mathbf{M}_2$ . Then we have

$$\operatorname{curl} (\mathbf{H}_2 - 4\pi\mathbf{M}_2) = 0$$

whence

$$\frac{1}{4\pi} \int \mathbf{H}_1 \cdot \mathbf{H}_2 d\tau = \int \mathbf{H}_1 \cdot \mathbf{M}_2 d\tau.$$

Similarly

$$W_m = \int \mathbf{H}_2 \cdot \mathbf{M}_1 d\tau.$$

Transformation to the expression

$$W_m = \iint \frac{|\mathbf{r} - \mathbf{r}'|^2 \mathbf{M}_1(\mathbf{r}) \mathbf{M}_2(\mathbf{r}') - 3(\mathbf{M}_1 \cdot (\mathbf{r}' - \mathbf{r}))(\mathbf{M}_2 \cdot (\mathbf{r}' - \mathbf{r}))}{|\mathbf{r} - \mathbf{r}'|^5} d\tau d\tau'$$

is only possible when the moment densities do not overlap.

7. Suppose that a current distribution is determined by a magnetization  $\mathbf{M}_z$  that is constant inside a sphere of radius  $R$  and zero outside. A simple calculation yields  $(8\pi/3)\mathbf{M}_z$  for the magnetic field at the centre (of course this is the field that is usually called  $\mathbf{B}$  in macroscopic theory). Since the value of this field does not depend on  $R$  it follows that a magnetization that is constant inside a spherical shell  $R_1 < r < R_2$  but zero outside produces no field at  $r = 0$ . Therefore if  $\mathbf{M}_z$  is a function of  $r$  we have

$$H(0) = \frac{8\pi}{3} M_z(0).$$

In the ground state of hydrogen we can write

$$\mathbf{i} = -c \operatorname{curl} \frac{e\hbar}{mc} |\psi|^2 \mathbf{s}$$

where  $\psi$  is the scalar Schrödinger wave function. Therefore

$$\mathbf{H}(0) = -\frac{8\pi}{3} \frac{e\hbar}{mc} |\psi(0)|^2 \cdot \mathbf{s}.$$



For the interaction with a nuclear moment  $(e\hbar/2Mc)g_i \cdot I$  it follows

$$\Delta U = \frac{8\pi}{3} \frac{e\hbar}{mc} \frac{e\hbar}{2Mc} |\psi(0)|^2 g_i (s \cdot I)$$

which is Fermi's well known formula.

8. The Hamiltonian for a system in a magnetic field is in reality a Hamiltonian for two coupled systems. As long as we are dealing with this complete system energies can be expressed in terms of magnetic fields. Introducing a vector potential is the price we have to pay for being able to eliminate the system that produces the field. We should bear in mind however, that the one particle Hamiltonian that is obtained in this way, determines the total energy. This is also obvious from the simple example discussed in section 5.

A special arrangement of two coupled systems is the basis of the so called Bohm paradox, where an electron is supposed to move outside a cylindrical core inside which there is a magnetic field parallel to the axis, but which has no stray field. The interaction between core and electron is a magnetic field energy, which can be expressed in the various ways described in section 6, but of which only the expression

$$\int \mathbf{i}_{el} \cdot \mathbf{A}_{core} d\tau$$

leads to a one particle Hamiltonian. Incidentally, it is rather obvious that the energy of the core is influenced by the field of the electron.

It is an essential feature of quantum mechanics that it is the energy of the complete system that determines emission frequencies, interference phenomena etc. For instance in hyperfine structure the frequencies of spectral lines are determined by the energy of nucleus plus electron. Most of the field energy,

$$\frac{1}{4\pi} \int \mathbf{H}_{el} \cdot \mathbf{H}_{nuc} d\tau$$

stems from a region close to the nucleus whereas the periodically varying charge density that leads to emission is much further out.

It is a merit of the Bohm paradox that it brings this fundamental feature forcefully to our attention.

# TEST OF ROLE OF STATISTICAL MODEL AT HIGH ENERGIES

S. D. DRELL

*Stanford Linear Accelerator Center, Stanford University*

and

D. R. SPEISER and J. WEYERS

*Centre de Physique Nucléaire, Université de Louvain*

*(Received June 10, 1965)*

The statistical model plays an unclear role at present in high-energy scattering events [1]. The low momentum transfer, or small angle scattering, events that comprise the bulk of the observed high-energy interactions find a natural and simple qualitative interpretation in terms of the peripheral model [2]. Both in baryon-baryon and meson-baryon inelastic collisions the vast majority of the secondary particles produced at high energies emerge into narrow forward or backward oriented cones about the collision axis and the transverse momentum transfer in the collision is  $\approx 300 \text{ MeV} \approx 2m_\pi c$ . The shadow of these dominant inelastic events leads to a diffraction cross section for elastic scattering which is also strongly peaked for low momentum transfers, or large impact parameters  $\gtrsim \hbar/2m_\pi c \sim 0.7 \times 10^{-13} \text{ cm}$ . In these glancing or peripheral collisions it is the component of the interaction with the longest range that controls the behavior of the participants in the collision.

It is when we turn to the central collisions that we anticipate the possibility that the concepts of the statistical model may find their natural application [3]. As in the low-energy nuclear physics domain (aside from the direct interaction processes) the colliding particles may be envisioned as forming a compound system with many channels leading to the various possible final state configurations. Aside from phase space and other kinematic factors, the various open reaction channels should be excited with equal probabilities and random relative phases in a statistical model.

This is the very basic general assumption underlying a statistical model. The statistical model has other characteristic predictions with regard to energy and angle variations of elastic cross sections and of multiplicities, in addition, for inelastic ones. These features, however, are tied to various models and "plausible" dynamical assumptions. Recently arguments have been put forward by Bethe and by Woo [1] pointing out the difficulty of reconciling the observed precipitous drop with energy of the large angle component of the elastic cross sections with the statistical model. It is at present not at all clear whether or not the experimental data should be interpreted as indicating the presence of a statistical component in high-energy collisions.

In this contribution that we are presenting to a leading pioneer in the development of the statistical model of nuclear reactions, we wish to propose a feasible program for testing the validity of the statistical model in high-energy collisions. The idea presented here is to check the very general premise of the statistical model that all open channels should contribute with equal probabilities and with random relative phases, independent of more detailed dynamical questions of specific energy or angle variation of the cross sections.

We consider two-body reactions involving incident meson or photon beams

$$\text{meson} + \text{baryon} \rightarrow \text{meson} + \text{baryon} \quad (1)$$

$$\text{photon} + \text{baryon} \rightarrow \text{meson} + \text{baryon}. \quad (2)$$

These reactions can proceed through many channels with different quantum numbers. It is the relative roles of channels of different angular momenta that control the angular and energy behaviour of cross sections, and of channels with different internal symmetry quantum numbers that determine the branching ratios for the production of final baryons and mesons with different charge or hypercharge quantum numbers. It is upon these branching ratios that we wish to focus attention.

The identification and enumeration of these channels are based on the octet model of the  $SU_3$  unitary symmetry group which has met with considerable success [4, 5]. In two-body reactions of types (1) or (2), the symmetry breaking mass splittings between particles belonging to the same  $SU_3$  multiplets may be expected to imply only small cor-

rections to the exact  $SU_3$  predictions if we consider experiments at high energies ( $s \equiv E_{CM}^2 \gg M^2$ ) and large angles, or momentum transfers ( $t \approx -\frac{1}{2}s$  at a  $90^\circ$  scattering angle in the center-of-mass frame) only. For such central collisions the statistical model should apply if it is at all valid in the realm of high-energy collisions. After they are averaged over energy and momentum transfer intervals large compared with the mass splittings within the individual multiplets ( $\Delta t^{\frac{1}{2}}, \Delta s^{\frac{1}{2}} > \Delta M$ ), the branching ratios should be determined solely by the combination coefficients, i.e., the appropriate Clebsch-Gordan coefficients, to form the different  $SU_3$  channels. That each  $SU_3$  channel contributes with equal amplitude and random phase is the very basic and the sole feature of the statistical model on which we base our predictions.

Independent of this model, there exist experimental tests of the accuracy of the role of  $SU_3$  itself in high-energy collisions. Levinson, Lipkin, and Meshkov [6] have derived the following equalities from  $SU_3$

$$\begin{aligned} d\sigma(K^- + P \rightarrow \pi^+ + \Sigma^-) &= d\sigma(K^- + P \rightarrow K^0 + \Xi^0) \\ d\sigma(\pi^- + P \rightarrow K^+ + \Sigma^-) &= d\sigma(K^- + N \rightarrow K^0 + \Xi^-) \end{aligned} \quad (3)$$

for reactions of type (1).

In general, simple equalities such as (3) do not emerge from the unitary symmetry model alone since there are a number of open channels through which the reaction can proceed and their relative phases and magnitudes require the input of dynamical assumptions. Formally, this is stated in the observation that both meson and baryon form octet representations in  $SU_3$  and their product can form 1, 8, 8', 10,  $\bar{10}$ , and 27 dimensional representations. The reaction can thus proceed through any of seven channels (including 8,8' mixing) and their relative amplitudes and phase factors at any energy determine the branching ratios. Therefore, analyses of these two-body reactions have heretofore contributed little to our confidence in  $SU_3$  which derives largely from its great success in classifying of multiplets and in predicting mass splittings within the individual multiplets. Moreover, the intensity of incident meson beams at high energies has been limited so that only a negligible number of events are observed in the laboratory under the condition of large  $t$  as desired to avoid large distor-



tions due to mass splittings and kinematic factors from the exact  $SU_3$  as a symmetry in high-energy scattering processes.

Assuming verification of relations (3) one may consider arbitrary reactions of type (1) and use the statistical model to make definite and unique predictions of the branching ratios for experimental testing.

Turning to reaction (2) we call attention to the important practical fact that a very intense current of 20 GeV electrons is anticipated at SLAC when operative and the resulting photon flux is of sufficiently high intensity to more than compensate for the appearance of a fine structure constant  $\alpha = 1/137$  in the ratio of the photon to meson cross sections, (2) to (1). Therefore, if the transformation properties of the electromagnetic current can be established in the unitary symmetry scheme, processes (2) may play a significant practical role in the testing of the statistical hypothesis for large  $s$  and  $t$  collisions.

In Lagrangian models of the  $SU_3$  symmetry scheme for elementary particles, it is most natural to introduce the electromagnetic current as a unitary octet [4, 5]. It is on this basis that we shall proceed in discussing the branching ratios in (2) with the statistical model. However, it is also possible for the electromagnetic current to have a unitary-singlet component and independent evidence on the transformation properties of the current is desired. The following relations between magnetic moments and between transition amplitudes have been proposed [5, 7] as tests of the assumption that the electromagnetic current is a pure octet and a  $U$ -spin scalar

$$\begin{aligned} \mu_N &= 2\mu_A \\ \langle \rho^0 | \eta \gamma \rangle &= \sqrt{3} \langle \rho^+ | \pi^+ \gamma \rangle. \end{aligned} \quad (4)$$

In calculating matrix elements, this is equivalent to equating a photon to the neutral member of the isotopic triplet,  $\rho^0$ , and to the isotopic singlet,  $\varphi$ , in the vector meson octet according to the relation.

$$|\gamma\rangle = \left\{ |\rho^0\rangle + \frac{1}{\sqrt{3}} |\varphi\rangle \right\}. \quad (5)$$

Practical results of these considerations are summarized in the following tables. The parameter  $\alpha$  appears as a mixing parameter for the two independent channels of a meson-nucleon system that transform as an octet. Denoting the corresponding states by  $|8\rangle$ ,

to which we assign the meson and nucleon octets, and  $|8'\rangle$ , respectively, we form the linear combinations

$$8_1\rangle = \cos \alpha |8\rangle - \sin \alpha |8'\rangle$$

$$8_2\rangle = \sin \alpha |8\rangle + \cos \alpha |8'\rangle.$$

The rotation angle  $\alpha$  is defined by the condition of orthogonality

$$\langle 8_1 | 8_2 \rangle = 0$$

and  $\alpha = 0$  if the additional symmetry of  $R$  invariance [4, 5, 8] is invoked. Whereas the consequences of  $R$  symmetry are unwelcome at low energies [9] it is possible that  $R$  may emerge as an approximate symmetry operation at high energies. If the special relations between cross sections that are independent of  $\alpha$  are verified by experiment and confirm the role of the statistical assumption in high energy central collisions, it will be possible to determine  $\alpha$  from the general ratios.

Table I – gives the ratio of differential cross sections at the same values of  $s$  and  $t$  for an incident  $\pi^+$  meson beam on a hydrogen target to form a meson 8 plus a baryon 8.

Table II – gives the ratios for a  $\pi^-$  + proton to form a meson 8 + baryon 8. Various ratios independent of mixing angle  $\alpha$  are also constructed.

Table III – gives the ratios for a  $\pi^+$  + proton to form a meson 8 + baryon 10.

Table IV – gives the ratios for a  $\pi^-$  + proton to form a meson 8 + baryon 10.

Tables V and VI – gives the corresponding ratios for photons incident on proton.

TABLE I

$$\begin{array}{l} \pi^+ \text{P} \rightarrow \pi^+ \text{P} \quad \frac{1}{2} \\ \quad \rightarrow \text{K}^+ \Sigma^+ \quad \frac{1}{2} \end{array}$$

TABLE II

$$\begin{aligned}
\pi^- P &\rightarrow \pi^- P & \frac{48}{225} + \frac{41}{450} \sin^2 2\alpha + \frac{\sqrt{5}}{75} \sin 4\alpha \\
&\rightarrow \pi^0 N & \frac{39}{225} + \frac{41}{900} \sin^2 2\alpha + \frac{\sqrt{5}}{150} \sin 4\alpha \\
&\rightarrow K^+ \Sigma^- & \frac{48}{225} - \frac{49}{450} \sin^2 2\alpha \\
&\rightarrow K^0 \Sigma^0 & \frac{39}{225} - \frac{49}{900} \sin^2 2\alpha \\
&\rightarrow K^0 \Lambda^0 & \frac{17}{150} + \frac{19}{300} \sin^2 2\alpha - \frac{\sqrt{5}}{150} \sin 4\alpha \\
&\rightarrow \eta N & \frac{17}{150} - \frac{11}{300} \sin^2 2\alpha - \frac{\sqrt{5}}{75} \sin 4\alpha \\
\frac{2[\pi^- P \rightarrow \pi^0 N] - [\pi^- P \rightarrow \pi^- P]}{2[\pi^- P \rightarrow K^0 \Sigma^0] - [\pi^- P \rightarrow K^+ \Sigma^-]} &= 1 \\
\frac{[\pi^- P \rightarrow \pi^- P] + [\pi^- P \rightarrow \eta N] + [\pi^- P \rightarrow K^0 \Sigma^0]}{[\pi^- P \rightarrow \pi^0 N] + [\pi^- P \rightarrow K^0 \Lambda^0] + [\pi^- P \rightarrow K^+ \Sigma^-]} &= 1 \\
\frac{[\pi^- P \rightarrow K^0 \Lambda^0] - [\pi^- P \rightarrow \eta N]}{[\pi^- P \rightarrow \pi^- P] - [\pi^- P \rightarrow K^+ \Sigma^-]} &= \frac{1}{2}
\end{aligned}$$

TABLE III

$$\begin{aligned}
\pi^+ P &\rightarrow \pi^+ N_{\frac{3}{2}}^{*+}(1238) & \frac{3}{16} \\
&\rightarrow \pi^0 N_{\frac{3}{2}}^{*++} & \frac{9}{32} \\
&\rightarrow K^+ Y_1^{*+}(1385) & \frac{3}{16} \\
&\rightarrow \eta N_{\frac{3}{2}}^{*++} & \frac{11}{32}
\end{aligned}$$

TABLE IV

$$\begin{aligned}
\pi^- P &\rightarrow \pi^+ N_{\frac{3}{2}}^{*-} & \frac{203}{1200} - \frac{2}{75} \sin^2 2\alpha + \frac{\sqrt{5}}{50} \sin 4\alpha \\
&\rightarrow \pi^0 N_{\frac{3}{2}}^{*0} & \frac{707}{7200} - \frac{4}{225} \sin^2 2\alpha + \frac{\sqrt{5}}{75} \sin 4\alpha \\
&\rightarrow \pi^- N_{\frac{3}{2}}^{*+} & \frac{61}{450} - \frac{2}{225} \sin^2 2\alpha + \frac{\sqrt{5}}{150} \sin 4\alpha \\
&\rightarrow K^+ Y_1^{*-} & \frac{203}{3600} - \frac{2}{225} \sin^2 2\alpha + \frac{\sqrt{5}}{150} \sin 4\alpha \\
&\rightarrow K^0 Y_1^{*0} & \frac{167}{1800} - \frac{1}{225} \sin^2 2\alpha + \frac{\sqrt{5}}{300} \sin 4\alpha \\
&\rightarrow \eta N_{\frac{3}{2}}^{*0} & \frac{11}{96} \\
\frac{2[\pi^- P \rightarrow K^0 Y_1^{*0}] - [\pi^- P \rightarrow \pi^- N_{\frac{3}{2}}^{*+}]}{[\pi^- P \rightarrow \eta N_{\frac{3}{2}}^{*0}]} &= \frac{24}{55}
\end{aligned}$$

$$\frac{[\pi^- P \rightarrow \pi^+ N_{\frac{3}{2}}^{*-}]}{[\pi^- P \rightarrow K^+ Y_1^{*-}]} = 3;$$

This is an exact result of  $SU_3$  independent of the statistical assumption [6].

$$\frac{6[\pi^- P \rightarrow K^0 Y_1^{*0}] - [\pi^- P \rightarrow \pi^+ N_{\frac{3}{2}}^{*-}]}{3[\pi^- P \rightarrow \pi^- N_{\frac{3}{2}}^{*+}] - [\pi^- P \rightarrow \pi^+ N_{\frac{3}{2}}^{*-}]} = \frac{3}{19}$$

$$\frac{2[\pi^- P \rightarrow \pi^+ N_{\frac{3}{2}}^{*-}] - 3[\pi^- P \rightarrow \pi^0 N_{\frac{3}{2}}^{*0}]}{[\pi^- P \rightarrow \pi^- N_{\frac{3}{2}}^{*+}] - [\pi^- P \rightarrow K^+ Y_1^{*-}]} = \frac{2}{38}$$

TABLE V

$$\begin{aligned} \gamma P &\rightarrow \pi^+ N & \frac{3}{2} \frac{4}{25} + \frac{1}{2} \frac{9}{25} \sin^2 2\alpha - 2 \frac{\sqrt{5}}{25} \sin 4\alpha \\ &\rightarrow \pi^0 P & \frac{1}{4} \frac{3}{50} + \frac{1}{4} \frac{9}{50} \sin^2 2\alpha - \frac{\sqrt{5}}{25} \sin 4\alpha \\ &\rightarrow K^+ \Sigma^0 & \frac{1}{4} \frac{3}{50} - \frac{1}{4} \frac{1}{50} \sin^2 2\alpha + 2 \frac{\sqrt{5}}{25} \sin 4\alpha \\ &\rightarrow K^0 \Sigma^+ & \frac{3}{2} \frac{4}{25} - \frac{1}{2} \frac{1}{25} \sin^2 2\alpha + 4 \frac{\sqrt{5}}{25} \sin 4\alpha \\ &\rightarrow K^+ \Lambda & \frac{3}{1} \frac{1}{50} + \frac{1}{1} \frac{1}{50} \sin^2 2\alpha - \frac{\sqrt{5}}{75} \sin 4\alpha \\ &\rightarrow \eta P & \frac{3}{1} \frac{1}{50} - \frac{3}{50} \sin^2 2\alpha \end{aligned}$$

$$\frac{2[\gamma P \rightarrow K^+ \Sigma^0] - [\gamma P \rightarrow K^0 \Sigma^+]}{2[\gamma P \rightarrow \pi^0 P] - [\gamma P \rightarrow \pi^+ N]} = 1$$

$$\frac{3[\gamma P \rightarrow \pi^0 P] - [\gamma P \rightarrow K^+ \Lambda] + 2[\gamma P \rightarrow \eta P]}{2[\gamma P \rightarrow \pi^0 P] - [\gamma P \rightarrow \pi^+ N]} = \frac{17}{7}$$

$$\frac{[\gamma P \rightarrow K^0 \Sigma^+] + [\gamma P \rightarrow \pi^0 P] + [\gamma P \rightarrow K^+ \Lambda]}{[\gamma P \rightarrow K^+ \Sigma^0] + [\gamma P \rightarrow \pi^+ N] + [\gamma P \rightarrow \eta P]} = 1$$

$$\frac{[\gamma P \rightarrow \pi^0 P] - [\gamma P \rightarrow K^+ \Sigma^0]}{[\gamma P \rightarrow K^+ \Lambda] - [\gamma P \rightarrow \eta P]} = 1$$

TABLE VI

$$\begin{aligned} \gamma P &\rightarrow \pi^+ N_{\frac{3}{2}}^{*0} & \frac{7}{50} + \frac{4}{2} \frac{1}{25} \sin^2 2\alpha + \frac{\sqrt{5}}{25} \sin 4\alpha \\ &\rightarrow \pi^0 N_{\frac{3}{2}}^{*+} & \frac{4}{1} \frac{1}{200} + \frac{8}{2} \frac{1}{25} \sin^2 2\alpha + 2 \frac{\sqrt{5}}{25} \sin 4\alpha \\ &\rightarrow \pi^- N_{\frac{3}{2}}^{*++} & \frac{4}{2} \frac{9}{50} + \frac{4}{75} \sin^2 2\alpha + \frac{\sqrt{2}}{75} \sin 4\alpha \\ &\rightarrow K^+ Y_1^{*0} & \frac{7}{100} + \frac{2}{2} \frac{1}{25} \sin^2 2\alpha + \frac{\sqrt{5}}{4} \frac{1}{50} \sin 4\alpha \\ &\rightarrow K^0 Y_1^{*+} & \frac{1}{6} \frac{9}{50} + \frac{4}{2} \frac{1}{25} \sin^2 2\alpha + \frac{\sqrt{5}}{2} \frac{1}{25} \sin 4\alpha \\ &\rightarrow \eta N_{\frac{3}{2}}^{*+} & \frac{1}{4} \frac{1}{8} \end{aligned}$$

$$\frac{2[\gamma P \rightarrow \pi^+ N_{\frac{3}{2}}^{*0}] - [\gamma P \rightarrow \pi^0 N_{\frac{3}{2}}^{*+}]}{[\gamma P \rightarrow K^0 Y_1^{*+}] - 2[\gamma P \rightarrow K^+ Y_1^{*0}]} = \frac{5}{34}$$



$$\frac{[\gamma P \rightarrow \pi^+ N_{\frac{3}{2}}^{*0}] + [\gamma P \rightarrow \pi^0 N_{\frac{3}{2}}^{*+}] - [\gamma P \rightarrow \pi^- N_{\frac{3}{2}}^{*++}]}{[\gamma P \rightarrow \pi^0 N_{\frac{3}{2}}^{*+}] - 2[\gamma P \rightarrow \pi^+ N_{\frac{3}{2}}^{*0}]} = \frac{17}{59}$$

$$\frac{[\gamma P \rightarrow \pi^+ N_{\frac{3}{2}}^{*0}]}{[\gamma P \rightarrow K^+ Y_1^{*0}]} = 2; \quad \text{This is an exact result of } SU_3 \text{ independent of the statistical assumption.}$$

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# VERTICES WITH PARTIAL $SU(6, 6)$ STRUCTURE

A. PAIS

*The Rockefeller Institute, New York*

*(Received June 10, 1965)*

## 1. INTRODUCTION

The role of the dynamical group  $SU(6)$  in the determination of effective vertex (three-point) structure has been much discussed lately, for strong, electromagnetic and weak phenomena. As the vertices for pseudoscalar and vector mesons coupled to baryons vanish when all three-momenta are zero, one needs to go beyond the static group  $SU(6)$ . For this purpose one must Lorentz transform (boost) supermultiplets to finite momenta. This is in general not a unique procedure, the same supermultiplet can be boosted in different ways. A book keeping which enumerates the variety of such boosts for a given  $SU(6)$ -representation can be made in terms of a non-compact group [1, 2, 3], the most symbol minded name [4] for which is  $SU(6, 6)$ .

One may attempt to compose effective  $n$ -point functions by means of the  $SU(6, 6)$  algebra, for all  $n$ . This leads to complications, however. First and foremost, there arises the now well known unitarity difficulty [5] which would be disastrous if  $SU(6, 6)$  were to be a strict symmetry, and which is still most uncomfortable even if the evidently approximate role of this, as of other big groups, is taken into consideration. Nor are the predictions for  $n > 3$  particularly convincing, perhaps with the possible exception of  $\bar{N}N$ -annihilation at rest [6]. It is therefore indicated to return to the analysis of the vertices, where the principal successes of  $SU(6)$  were found in the first place, and to reexamine the conditions under which these encouraging results were obtained. An attempt in this direction is made in the present note.

It is well to recall that most of the promising answers refer to non-relativistic, if not static quantities. Up to and including the first order in  $q$ , the momentum transfer, things look pretty good [7]. But even for the vertices,  $SU(6, 6)$  does not fare so well to order  $q^2$  and higher.

In particular, the prediction that the Sachs charge form factor of the proton,  $F_{\text{ch}}^p(q^2)$ , rises as  $q^2$  compared to the Sachs magnetic form factor  $F_{\text{mag}}^p(q^2)$  is not a good one. Thus a reasonable point of departure appears to be to ask the question to what extent the results to order  $q$  determine the results to order  $q^2$  and up, inasfar as vertices are concerned. For this purpose it appears interesting to study the consequences of the following assumptions.

(1) In the sense of  $SU(6, 6)$  the baryons appear in the vertex in the  $364$ -representation, that is, the totally symmetric structure of the  $56$  (static  $SU(6)$ ) is maintained for non-zero momenta.

(2) The bilinear baryon charge and current densities which enter in the vertices have the structure of  $143$ , in the sense of  $SU(6, 6)$ .

(3) Beyond this, *only the usual requirements of Lorentz invariance and  $SU(3)$  are made.*

Thus only a partial  $SU(6,6)$  structure is imposed, namely with regard to the baryon densities [8]. It is not asked that the mesons (as they enter in the BBM vertices) are in the  $143$  of  $SU(6, 6)$ , much less that the strong vertex is an  $SU(6, 6)$  scalar. It is required though that the vertex is a Lorentz covariant  $SU(3)$  scalar. (Questions of breakdown of  $SU(3)$  lie beyond the scope of this note.) It is not asked that the electromagnetic baryon vertex behaves as  $143$  with respect to  $SU(6, 6)$ , but only as  $8$  with respect to  $SU(3)$ : and similarly for semileptonic vertices.

Under these conditions the good results to order  $q$  are maintained.

It must now further be asked if this partial  $SU(6, 6)$  structure has consequences to order  $q^2$  and up. The following will be shown.

(A) For pseudoscalar and/or pseudovector couplings

$$\frac{D}{F} = \frac{3}{2} \text{ for all } q^2. \quad (1.1)$$

(B) Let  $F_{\text{ch}}(q^2)$  be the charge form factor for any member of the baryon octet.  $F_{\text{ch}}(0) = Q$  the charge of that baryon. Then

$$F_{\text{ch}}(q^2) = \frac{Q}{Q(p)} F_{\text{ch}}^p(q^2), \text{ for all } q^2. \quad (1.2)$$

This applies also the  $(\Sigma^0|A)$ -transition form factor (where  $Q = 0$ , of course).

(C) Likewise, for any member of the baryon octet as well as for the  $(\Sigma^0|A)$  transition case,

$$F_{\text{mag}}(q^2) = \frac{M}{M(p)} F_{\text{mag}}^p(q^2), \text{ for all } q^2. \quad (1.3)$$

$F_{\text{mag}}(0) = M$ , the magnetic moment. For the neutron, it is sufficient to obtain (1.2, 3) to require that the nucleon densities have  $SU(4, 4)$  structure. Here  $SU(4, 4)$  is related to the zero hypercharge subgroup [9]  $SU(4)$  of  $SU(6)$  just as  $SU(6, 6)$  is related to  $SU(6)$ . (On the other hand,  $M(n)/M(p) = -\frac{2}{3}$  does not follow from  $SU(4)$  unless additional assumptions are made [10].) The neutron relations were first written down by Barnes, Carruthers and von Hippel [11] under more restrictive assumptions. These relations are in qualitative agreement with experiment over the known range of  $q^2$ .

With the baryons on the mass shell, no assumptions need to be made in the present case about analytic continuation beyond the usual ones for a Lorentz covariant vertex.

It will also be shown (Section 5) that the relations (1.2, 3) may be considered to hold *without* having imposed symmetry conditions which are violated by the kinetic energy.

The further general discussion is given in Sec. 5. In Sec. 2 a brief discussion is given of some properties of boost matrices, in Sec. 3 Lorentz transformations of supermultiplets are reviewed, while in Sec. 4 a short derivation is given of the vertex structure. It is hoped that Secs. 2-4 which contain many known results may help to lighten somewhat the formal apparatus.

## 2. BOOST MATRICES

The metric will be  $p_\mu = (p, ip_0)$ . The Dirac matrices  $\gamma_\mu$ ,  $\mu = 1, \dots, 4$  will be taken hermitian. We shall use the representation  $\gamma_4 = \rho_3$ ,  $\gamma = \rho_2 \sigma$ ; and  $\gamma_5 = \rho_1$ . Let  $D_i^\alpha(p)$  denote the set of solutions of the free Dirac equation for mass  $m$ .  $\alpha = 1, \dots, 4$  numbers the components.  $D_i^\alpha = (u_1^\alpha, u_2^\alpha, v_1^\alpha, v_2^\alpha)$ ,  $u_i^\alpha$  are particle solutions for  $p, p_0$ ,  $v_i^\alpha = (\gamma_5 u)_i^\alpha$  are solutions with  $-p, -p_0$ , ( $p_0 > 0$ ). We have ( $\gamma p \equiv \gamma_\mu p_\mu$ )

$$D_i^\alpha(p) = \left[ \frac{m - i(\gamma p)\gamma_4}{\{2m(p_0 + m)\}^{\frac{1}{2}}} \right]_i^\alpha, \quad (2.1)$$



$$D_i^\alpha(0) = \delta_i^\alpha \quad \text{and} \quad D_i^\alpha(p) = D_\beta^\alpha(p) D_i^\beta(0). \quad (2.2)$$

$D_\beta^\alpha(p)$  is the boost matrix, a linear transformation on the zero momentum components which generates the solutions for momentum  $p$ .

$D_i^\alpha$  is a matrix with one index in component space, one in state space. The boost matrix  $D_\beta^\alpha$  is the *same* matrix but with both indices in component space. Depending on the nature of the indices we have a different meaning for the adjoint. As usual,  $\bar{D}_\beta^i(p) = D_\alpha^{\dagger i}(\gamma_4)_\beta^\alpha$ , for all  $p$ . We define  $\bar{D}_\beta^\alpha$  by taking the adjoint of Eq. (2.2)

$$\bar{D}_\alpha^i(p) = \bar{D}_\beta^i(0) \bar{D}_\alpha^\beta(p), \quad (2.3)$$

so that

$$\bar{D}_\alpha^\beta(p) = (\gamma_4 D(p) \gamma_4)_\alpha^\beta. \quad (2.4)$$

Observe that we may read eqs. (2.2) and (2.4) as

$$D^\alpha(p) = D_\beta^\alpha(p) D^\beta(0), \quad \bar{D}_\alpha(p) = \bar{D}_\beta(0) \bar{D}_\alpha^\beta(p) \quad (2.5)$$

for any state  $i$ .

From now on the matrices  $D$ ,  $\bar{D}$  will always be the boost matrices in component space. Note the following properties:

$$D\bar{D} = 1 \quad (2.6)$$

$$D\gamma_5\bar{D} = \gamma_5 \quad (2.7)$$

$$D\gamma_4\bar{D} = -i(\gamma p)/m \quad (2.8)$$

$$D\gamma\varepsilon(0)\bar{D} = \gamma\varepsilon(p) \quad (2.9)$$

where

$$\begin{aligned} \varepsilon_\mu(p) &= (\varepsilon(p), i\varepsilon_0(p)), \\ \varepsilon(p) &= \varepsilon(0) + \frac{p(p\varepsilon(0))}{m(p_0 + m)}, \\ \varepsilon_0(p) &= \frac{p\varepsilon(0)}{m}, \end{aligned} \quad (2.10)$$

so that

$$p\varepsilon(p) = 0. \quad (2.11)$$

From (2.8, 9, 11)

$$D\gamma_4 \gamma \varepsilon(0) \bar{D} = \sigma_{\mu\nu} p_\mu \varepsilon_\nu(\mathbf{p})/m, \quad (2.12)$$

$$\sigma_{\mu\nu} = -i[\gamma_\mu, \gamma_\nu]/2.$$

Finally let  $C$  be the charge conjugation matrix,  $C^{-1}\gamma_\mu C = -\gamma_\mu^t$ ,  $C^t = -C$ ,  $t$  is transpose. Then

$$CD^t = \bar{D}C, \quad \bar{D}^t C^{-1} = C^{-1}D. \quad (2.13)$$

In the chosen representation  $C$  may be taken as

$$C = i\gamma_5 \sigma_2. \quad (2.14)$$

### 3. BOOSTED SUPERMULTIPLETS

Let

$$M = -iP - \sigma \varepsilon(0)V$$

denote the meson matrix for the 35-representation of SU(6).  $P$  is the ps octet,  $V$  the vector nonet,  $\varepsilon(0)$  the polarization vector. All SU(3) and spin state labels are suppressed. Note, however, that for  $\mathbf{p} = 0$ ,  $M$  acts on spin states  $\chi_i^\alpha = \delta_i^\alpha$ ,  $\alpha, i = 1, 2$ . Because of the symmetry in  $\alpha$  and  $i$ , we may look upon  $\sigma$  as a matrix in component space.  $\text{Tr}(M^\dagger M)$  is the invariant bilinear form of SU(6).

One may multiply  $M$  by anything that is spin and unitary spin independent and still have SU(6) structure (which always refers to  $\mathbf{p} = 0$ ). Define

$$^{(1)}\mathcal{M}(0) = \gamma_5 \gamma_4 M, \quad (3.1)$$

$$^{(2)}\mathcal{M}(0) = \gamma_5 M. \quad (3.2)$$

As  $\gamma_5$  and  $\gamma_5 \gamma_4$  commute with  $\sigma$  we still have SU(6) structure but we have doubled the number of rows and columns. Define

$$^{(i)}\mathcal{M}(\mathbf{q}) = D(\mathbf{q}, \mu)^{(i)} \mathcal{M}(0) \bar{D}(\mathbf{q}, \mu); \quad i = 1, 2 \quad (3.3)$$

and use (2.6-12):

$$^{(1)}\mathcal{M}(\mathbf{q}) = i\gamma \varepsilon(\mathbf{q})V - i\gamma_5 \frac{(\gamma q)}{\mu} P,$$

$$^{(2)}\mathcal{M}(\mathbf{q}) = \frac{-i\sigma_{\mu\nu} q_\mu \varepsilon_\nu(\mathbf{q})}{\mu} V - i\gamma_5 P$$

which are the two boosted meson matrices introduced earlier [12].

Define

$$\begin{aligned} \mathcal{M}(q) = & f_V(q^2) \cdot i\gamma_5(q)V - f_A(q^2) \cdot i\gamma_5 \frac{(\gamma q)}{\mu} P + \\ & -f_T(q^2) \cdot \frac{i\sigma_{\mu\nu} q_\mu \varepsilon_\nu(q)}{\mu} - f_P(q^2) \cdot i\gamma_5 P, \end{aligned} \quad (3.4)$$

with four independent weight functions  $f_V, f_A, f_T, f_P$ . These functions are constants on (but not off) the mass shell.  $\mathcal{M}(q)$  is the most general way [13] the  $SU(3)$  meson multiplets can enter a vertex. Note that [12]

$$\begin{aligned} \mathcal{M}(q) = & f_V(q^2)^{(1)} \mathcal{M}(q) + f_T(q^2)^{(2)} \mathcal{M}(q), \\ \text{if } f_V = f_A, \quad f_T = f_P \text{ all } q^2, \end{aligned} \quad (3.5)$$

and that

$$f_V = f_A = f_T = f_P, \text{ all } q^2, \text{ corresponds to } SU(6,6) \text{ structure of } \mathcal{M}(q). \quad (3.6)$$

Likewise we go from the well known  $SU(6)$  form of the baryon states [14] to the enlarged form  $B^{\alpha\beta\gamma, ABC}(0)$  at zero momentum, as follows. ( $A, B, C$  are  $SU(3)$  indices).

$$\begin{aligned} B^{\alpha\beta\gamma, ABC}(0) = & \chi^{(\alpha\beta\gamma)} d^{ABC} + \frac{1}{3\sqrt{2}} [e^{\alpha\beta} u^\gamma(0) X^{ABC} + e^{\beta\gamma} u^\alpha(0) X^{BCA} \\ & + e^{\gamma\alpha} u^\beta(0) X^{CAB}], \quad (3.7) \\ X^{ABC} = & \varepsilon^{ABC} b_D^C. \end{aligned}$$

As in Eq. (2.5), the spin *state* labels are suppressed. Further see (2.14),

$$\varepsilon^{\alpha\beta} = \left( \frac{1+\gamma_4}{2} \gamma_5 C \right)^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.8)$$

which is the old [14]  $\varepsilon^{ij}$  bordered by zeros. The spin  $\frac{3}{2}$  wave functions are

$$\chi_{ijk}^{(\alpha\beta\gamma)} = \frac{1}{6} \sum D_i^{(\alpha}(0) D_j^{\beta}(0) D_k^{\gamma)}(0) \quad (3.9)$$

The summation is over all permutations of  $\alpha, \beta, \gamma$ . Then

$$B^{\alpha\beta\gamma, ABC}(p) = D_\alpha^\alpha D_\beta^\beta D_\gamma^\gamma B^{\alpha'\beta'\gamma', ABC}(0). \quad (3.10)$$

In particular

$$e^{\alpha\beta} u^\gamma(0) \rightarrow \frac{1}{2} \left[ \left( 1 - \frac{i\gamma p}{m} \right) \gamma_5 C \right]^{\alpha\beta} u^\gamma(p), \quad (3.11)$$

see Eqs. (2.7, 8, 13). Eq. (3.8) describes a totally symmetric boost. Instead of operating with three  $D$ 's in (3.10) one could have taken some  $D$ 's and some  $\bar{D}$ 's. This gives rise to the non symmetric alternative boosts [15]; they are not used here.

#### 4. VERTEX STRUCTURE

Consider the baryon octet part of the vertex

$$\bar{B}(p_1) \mathcal{M}(q) B(p_2) \equiv \bar{B}_{\alpha\beta\gamma, ABC}(p_1) \mathcal{M}_{\delta D}^{\gamma C}(q) B^{\alpha\beta\delta, ABD}(p_2), \quad (4.1)$$

where  $q = p_2 - p_1$ .  $\mathcal{M}$  is given by the general form (3.4),  $B$  by (3.10). SU(3) contractions reduce this to

$$\begin{aligned} 4(D+F)\bar{u}\mathcal{M}u \cdot \left( 1 + \frac{q^2}{4m^2} \right) - \frac{1}{4}(D-F-2T)\bar{u}\{Z_2 Z_1 \mathcal{M}\}u + \\ - \frac{1}{4}(2D-2T)\bar{u}Z_2 \mathcal{M}' Z_1 u, \\ Z_i = 1 - \frac{i\gamma p_i}{m}, \quad \mathcal{M}' = \gamma_5 C \mathcal{M}^t C^{-1} \gamma_5, \end{aligned} \quad (4.2)$$

with the SU(3)-conventions

$$\begin{aligned} D\bar{u}\mathcal{M}u &= \bar{u}_B^A (\mathcal{M}u + u\mathcal{M})_A^B, \\ F\bar{u}\mathcal{M}u &= \bar{u}_B^A (\mathcal{M}u - u\mathcal{M})_A^B, \\ T\bar{u}\mathcal{M}u &= \bar{u}_B^A \mathcal{M}_C^B u_A^C. \end{aligned} \quad (4.3)$$

In the second line of Eq. (4.2),  $\{\}$  denotes the Dirac trace. A bit of  $\gamma$ -algebra yields (dividing through by 6 and dropping the  $\bar{u}$  and  $u$  symbols)

*P-vertex*

$$F_0(q^2) = -i \left( D + \frac{2F}{3} \right) \left( 1 + \frac{q^2}{4m^2} \right) \gamma_5 \left( \frac{2m}{\mu} f_A(q^2) + f_P(q^2) \right). \quad (4.4)$$

*V-vertex*



$$i[\gamma_\mu F_1(q^2) + \sigma_{\mu\nu} q_\nu F_2(q^2)]\epsilon_\mu, \quad (4.5)$$

$$F_1(q^2) = \left(F + T + \frac{q^2}{4m^2} \left(D + \frac{2F}{3} - \frac{T}{3}\right)\right) f_V(q^2) + \frac{q^2}{2m\mu} \left(D - \frac{F}{3} - \frac{4T}{3}\right) f_T(q^2), \quad (4.6)$$

$$2mF_2(q^2) = \left(D - \frac{F}{3} - \frac{4T}{3}\right) f_V(q^2) + \frac{2m}{\mu} \left(\left(D + \frac{2F}{3} - \frac{T}{3}\right) + \frac{q^2}{4m^2} (F + T)\right) f_T(q^2). \quad (4.7)$$

Define the corresponding Sachs type form factors

$$F_{\text{ch}} = F_1 - \frac{q^2}{2m} F_2, \quad F_{\text{mag}} = \frac{F_1}{2m} + F_2. \quad (4.8)$$

Then

$$F_{\text{ch}}(q^2) = \left(1 + \frac{q^2}{4m^2}\right) \left[f_V(q^2) - \frac{q^2}{4m^2} \cdot \frac{2m}{\mu} f_T(q^2)\right] (F + T), \quad (4.9)$$

$$F_{\text{mag}}(q^2) = \frac{1}{2m} \left(1 + \frac{q^2}{4m^2}\right) \left[f_V(q^2) + \frac{2m}{\mu} f_T(q^2)\right] \left(D + \frac{2F}{3} - \frac{T}{3}\right). \quad (4.10)$$

## 5. CONCLUSIONS

1) To order  $q$ , all four form factors in (3.4) are to be replaced by  $f(q^2) \rightarrow f(-\mu^2)$ . Eqs. (4.4, 9, 10) then contain all the usual non relativistic results regarding strong, electromagnetic and weak vertices. The one exception is the relation [9]  $g_A = 5g/3$ . In order to get this relation it is sufficient to have the relations (3.5) for  $q^2 = -\mu^2$  only [16].

2) For arbitrary  $q^2$  the form factors  $F_0$ ,  $F_{\text{ch}}$  and  $F_{\text{mag}}$  have the remarkable property (which evidently goes beyond  $SU(3)$ ) that in each of them the  $SU(3)$  dependence (that is, the dependence on  $D$ ,  $F$ ,  $T$ ) is the same for all  $q^2$ . This is true for the most general set of form factors in (3.4).

Eq. (1.1) follows directly from Eq. (4.4). Eqs. (4, 9, 10) describe properties of vector meson vertices. With the assumption that the electromagnetic couplings are proportional to the  $(\rho^0 + \omega^0/\sqrt{3})$ -coupling, Eqs. (1.2, 3) follow.

3) The usual SU(6, 6) conclusions about electric and magnetic form factors are found by inserting (3.6) in (4.9, 10). It may also be seen from (4.9, 10) that an alternative [17] SU(6, 6) with  $f_V = f_A = -f_T = -f_P$  is physically distinguishable from the SU(6, 6) defined by (3.6).

4) Consider the  $V$ -couplings

$$X_\mu \varepsilon_\mu V, \quad Y_\mu \varepsilon_\mu V \quad (5.1)$$

where

$$X_\mu = \frac{P_\mu}{2m}, \quad Y_\mu = \varepsilon_{\mu\nu\rho\sigma} \frac{P_\nu q_\rho}{2m \mu} \gamma_\sigma \gamma_5, \quad (5.2)$$

with  $P_\mu = p_\mu^1 + p_\mu^2$ . These couplings are actually comprised in (3.4), with

$$X_\mu \leftrightarrow f_V = 1, \quad f_T = -\frac{\mu}{2m}, \quad (5.3)$$

$$Y_\mu \leftrightarrow f_V = \frac{q^2}{2m\mu}, \quad f_T = 1. \quad (5.4)$$

$X_\mu$  and  $Y_\mu$  are two linearly independent forms ( $X_\mu Y_\mu = 0$ ), each with the property

$$[X_\mu, \gamma p^i] = [Y_\mu, \gamma p^i] = 0, \quad i = 1, 2, \quad (5.5)$$

that is, the couplings (5.1) commute with the baryon kinetic energy. This shows that (4.9, 10) may be considered to hold without violation by the kinetic energy by taking any linear combination of the couplings (5.1).

5) It follows from (5.3) that the particular linear combination

$$(X_\mu + Y_\mu) \varepsilon_\mu V \quad (5.6)$$

yields the additional relation

$$F_{\text{ch}}(q^2) = \frac{Q}{M} F_{\text{mag}}(q^2), \quad \text{all } q^2 \quad (5.7)$$

for all members of the baryon octet as well as for the ( $\Sigma^0 | A$ ) transition. Eq. (5.6) is the coupling condition proposed by Barnes, [18] who has emphasized [19] the possible importance of the proton relation (5.7).

6) The present formulation therefore appears to be of some use for two reasons. First, the conditions stated in Sec. 1 make clear that many results can be maintained under weaker requirements than was sometimes stated. Secondly, it is possible to judge the implications of more specific dynamical models by simply asking whether and how they can be expressed in terms of conditions on the four general form factors which appear in Eq. (3.4). The answers can then be read off from equations like (4.4–10). For example, a relation between  $M(p)$  and the proton charge radius [20] can be expressed in terms of a more specified structure [21] for  $f_V$  and  $f_T$ .

7) It is interesting that also the results for  $\bar{N}N$ -annihilation at rest are likewise essentially dependent only [22] on the same assumptions (1), (2) and (3) of Sec. 1.

8) An important aspect (and maybe a major shortcoming) of our present thinking about internal symmetries like isospin,  $SU(3)$  is that we can imagine a fictitious world in which these symmetries are exact without apparent strain on either any general postulates or the dynamics of strong interactions. Not so for  $SU(6)$  and its sequels. While quantities like rest mass, magnetic moment, coupling constants are all zero-(or low)-energy parameters, their effective values are code-termined by high virtual frequency contributions... one is led to surmise that, wherever an  $SU(6)$  prediction works well, there is a strong effective damping involved in these high-energy contributions. [23] This prelude to deeper dynamics may perhaps indicate further where and how this damping should be manifest. It is also hoped that this note may help explain to a good friend how the whole thing stands.

I gratefully acknowledge numerous discussions with Drs. M. A. B. Bég and N. Cabibbo.

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# COHERENT HIGH ENERGY REACTIONS WITH NUCLEI

ALFRED S. GOLDBABER\*

*Department of Physics, University of California,  
Berkeley, California*

and

MAURICE GOLDBABER

*Brookhaven National Laboratory,\*\* Upton, New York  
(Received June 16, 1965)*

## 1. INTRODUCTION

We present here a brief review of various notions on the use of atomic nuclei as targets in high energy elementary particle physics. While most of these notions are well known, we hope that a unified presentation will stimulate further thought on the subject. Our main concern is with coherent reactions, in which the target nucleus remains intact [1]. In this respect, we supplement the text of Blatt and Weisskopf, who speak of coherent scattering from nothing smaller than a molecule [2].

## 2. APPLICATIONS OF COHERENT REACTIONS

For ease of conception, let us begin with a spinless nucleus, such as  $\text{Ge}^{72}$ . For a coherent reaction which does not excite or break up the Ge,



we can immediately obtain some simple selection rules, if X is a  $\pi$  or K meson [3]. Since the only particle with spin (if any) is the product Y, the reaction amplitude for production in the forward direction takes the form

$$M = T_{i_1 i_2 \dots i_J} p_{i_1} p_{i_2} \dots p_{i_J} f(s, t) \quad (2)$$

\* Miller Institute Fellow.

\*\* Under the auspices of the U.S. Atomic Energy Commission.

where  $T_{i_1 i_2 \dots i_J}$  is a symmetric traceless  $J$ -index tensor representing the spin of  $Y$ , and  $\mathbf{p}$  is the momentum, measured in the  $Y$  rest frame, of one of the other three particles in the reaction. (For forward production these momenta are collinear.) Finally,  $s$  and  $t$  are the usual invariant energy and momentum transfer variables. The parity of the matrix element is simply  $(-1)^J$  from the number of momentum factors present. This means that a  $Y$  particle produced in the forward direction must have a parity  $P_Y = (-1)^J P_{\pi, K} = -(-1)^J$ , if parity is conserved in the reaction. For production away from the forward direction, there is a matrix element of opposite parity,

$$M' = T_{i_1 i_2 \dots i_J} (\mathbf{p}_{\text{Ge}_1} \times \mathbf{p}_{\text{Ge}_r})_{i_1} p_{i_2} p_{i_3} \dots p_{i_J} f'(s, t) \quad (3)$$

and the selection rule fails. However, we may expect the contribution of the  $M'$  term to the reaction to be quite small: The radius of  $\text{Ge}^{72}$  is about 4.5 fm, and the coherent production cross section should fall rapidly as the squared momentum transfer  $|t|$  exceeds  $(4.5 \text{ fm})^{-2}$ , or  $0.002 (\text{GeV})^2$ . (The argument for this momentum transfer dependence is given in section 3, below.). Since this is a scale of momentum transfer much smaller than the typical scale for production reactions on a single nucleon  $[0.01-0.02(\text{GeV})^2]$ , it is reasonable to expect that  $M'$ , which is zero in the forward direction, will not become appreciable before the whole amplitude is cut off by the coherency requirement. Therefore, coherent production of spin  $J$  mesons should obey the approximate selection rule

$$P_Y = -(-1)^J. \quad (4)$$

The requirement of small momentum transfer also gives a lower limit to the momentum of the incident beam  $X$  required for production of a particle with mass  $m_Y \neq m_X$ . The desired relation is easily derived:

$$p_{X \text{ lab}} \geq (m_Y^2 - m_X^2)(2q)^{-1}. \quad (5a)$$

Here  $q$  is the (small) momentum transfer to the nucleus. Inserting the requirement  $q \sim R^{-1}$  leads to an *approximate* threshold condition for coherent production,

$$p_{X \text{ lab}}(\text{threshold}) \approx 2.5(m_Y^2 - m_X^2)R \quad (5b)$$

where momentum and mass are measured in GeV, and  $R$  is measured in fermis.

In general, the value of experiments with a spin zero target is to reduce the number of variables and thus simplify analysis of the reaction. Another case in which this might be useful is high energy elastic proton-proton scattering. Foley *et al.* [4] obtained the real part of the forward scattering amplitude from the interference of the strong-interaction amplitude and the Coulomb amplitude at small angles. However, the analysis required the assumption of no spin dependence in the strong amplitude. Without this assumption, the possibility remained that the ratio of real to imaginary forward amplitude was zero instead of 0.3. With Ge as a target, this ambiguity would disappear, although at the price of introducing a new problem, the relation of results for Ge to results for H targets. We shall return to this difficulty later.

The reason for using Ge in the above examples lies in the possibility of making a counter-target of lithium-drifted germanium. The counter would be quite sensitive to the excitation or breakup of a Ge nucleus and thus could be used to select coherent events. The fast products of the reaction would be observed by appropriate detectors downstream from the target [5].

On the other hand, a useful tool for bubble chamber investigations using coherent reactions would be a  $\text{Ne}^{20}\text{-H}_2$  bubble chamber [6]. As the ratio of Ne to H is increased, the ratio of production at small angles of coherently "forbidden" to "allowed" products should fall steadily, giving a dramatic demonstration of the nuclear effects.

As we shall see in section 3, below, the rules derived for a spinless nucleus should also hold for a nucleus with spin, to order  $A^{-1}$ , where  $A = Z + N$  is the mass number. Accepting this, a possible example of the selection rule (4) has been observed by Allard *et al.* [7] for 16 GeV  $\pi^-$  in a freon ( $\text{C}_2\text{F}_5\text{Cl}$ ) bubble chamber. In this experiment, a considerable number of  $Y = A_1(1090) 3\pi$  events were observed, but  $Y = A_2(1310) 3\pi$  events were rarer by at least a factor of 4, and inseparable from background. The supposed quantum numbers of  $A_1$  and  $A_2$  are  $J^P = 1^+$  and  $2^+$ , respectively. Thus, the  $A_2$  would be suppressed by the selection rule. In contrast to the freon results, experiments with hydrogen targets show a ratio  $R_H = A_2/A_1$  ranging

from 2 or 3 at 3 GeV to 1 or 2 at 8 GeV [8]. Therefore, unless  $R_H$  changes considerably from 8 to 16 GeV, we have here an example of coherent production selection rules. Clearly, further work is required to verify this interpretation [9].

One may also consider coherent scattering from nuclei with zero isospin ( $N = Z$ ), such as  $D^2$ ,  $He^4$ ,  $Ne^{20}$ ,  $Ca^{40}$ . The interest in using such targets is, again, that the number of variables is reduced. For example, elastic scattering of  $K^+$ ,  $K^-$  from such a substance could be used to obtain precisely the forward scattering amplitudes for  $K^0$ ,  $\bar{K}^0$ , which would be useful in  $K_1$ - $K_2$  regeneration studies with the same substance as the regenerating medium [10].

### 3. DERIVATION OF COHERENT REACTION AMPLITUDES, AND FURTHER CONSEQUENCES

While a complete treatment of elementary particle reactions with nuclei in terms of reactions with nucleons has not been achieved, a qualitative discussion is illuminating [11]. Let us start with the unrealistic single-interaction approximation, in which the amplitude for reaction with the nucleus is obtained by summing the amplitudes for interaction with each of the nucleons in the nucleus. The matrix element in spin and isospin space for the  $i^{\text{th}}$  nucleon may be written

$$M_i = a + b \cdot \sigma_i + c_\alpha \tau_{\alpha i} + d_\alpha \cdot \sigma_i \tau_{\alpha i} \quad (6)$$

where  $\sigma_i$  is the Pauli spin vector, and  $\tau_{\alpha i}$ , the  $\alpha$  component of the Pauli isovector operator. The coefficients  $a \dots d_\alpha$  depend on the variables  $s$  and  $t$  for the reaction  $X + N_i \rightarrow Y + N_i$ , and on the spins, isospins and momentum directions of  $X$  and  $Y$ . In the single-interaction approximation, the matrix element for  $X + \mathcal{N} \rightarrow Y + \mathcal{N}$ , where  $\mathcal{N}$  represents the nucleus, is

$$\mathcal{M} = \langle \mathcal{N} | \sum_i M_i e^{iq \cdot R_i} | \mathcal{N} \rangle \quad (7)$$

with  $q$  the momentum transfer, and  $R_i$  the position of the  $i^{\text{th}}$  nucleon. Now, for every nucleon with spin up in a given direction, except one (odd-even nucleus) or two (odd-odd), there is another of the same charge with spin down. Thus, the  $b$  and  $d$  terms in  $M_i$  will make no contribution to  $\mathcal{M}$  for even-even nuclei, and at most a contribution of order  $A^{-1}$  for other nuclei. A requirement for coherency is that



the charge state of the  $i^{\text{th}}$  nucleon not change in  $M_i$ . Thus, the only contribution of  $c_\alpha \tau_{\alpha i}$  is from the diagonal element  $c_3 \tau_{3i}$ , where  $\tau_{3i}$  is  $+1$  for protons and  $-1$  for neutrons. Using this, we may write  $\mathcal{M}$  to order  $A^{-1}$  as

$$\begin{aligned}\mathcal{M} &\approx Aa\langle\mathcal{N}|e^{i\mathbf{q}\cdot\mathbf{R}}|\mathcal{N}\rangle + Zc_3\langle\mathcal{N}|e^{i\mathbf{q}\cdot\mathbf{R}_p}|\mathcal{N}\rangle + \\ &\quad - Nc_3\langle\mathcal{N}|e^{i\mathbf{q}\cdot\mathbf{R}_n}|\mathcal{N}\rangle \\ &= Aaf(q^2) + Zc_3f_p(q^2) - Nc_3f_n(q^2).\end{aligned}\quad (8)$$

Here,  $f(q^2)$  is the Fourier transform of the nucleon density distribution ( $f(0) = 1$ ), and  $f_p$  and  $f_n$  are the same quantities for protons alone and neutrons alone, respectively. The qualitative nature of these form factors may be seen in the idealized case of a uniform nucleus with radius  $R$ . The form factor here is

$$f(q^2) = \frac{3}{(qR)^3} [\sin qR - qR \cos qR] \quad (9)$$

yielding a diffraction pattern which does fall rapidly as  $qR$  exceeds unity, as mentioned earlier.

Relation (8) would be a good approximation if the forces were weak, but they are not. A total cross section of more than 20 mb for the interaction of X or Y with a nucleon implies a mean free path in nuclear matter of no more than 4 fm. It is plausible to assume that the effect of this high total interaction probability may be represented by a spin-and isospin-independent complex optical potential for the motion of X or Y through the nucleus [11]. Since the cross section for  $X \rightarrow Y$  at high energies is much less than the total (if  $X \neq Y$ ), we may still assume that  $\mathcal{M}$  is related to  $M_i$  approximately as before, except that the optical absorption suppresses the contribution of a channel through the middle of the nucleus. In the limit of high absorption, one may approximate the region for coherent production by a ring of radius  $R$ , containing a fraction  $\varepsilon$  of the nuclear material. The effective form factor becomes

$$f(q^2) = \varepsilon J_0(q_\perp R) \quad (10)$$

where  $q_\perp$  is the momentum transfer perpendicular to the incident direction, and  $J_0$  is the Bessel function of order zero. Again, of course,

there is a diffraction pattern with intensity falling rapidly as  $qR$  exceeds unity.

If the distributions of neutrons and protons had different radii, the absorption would increase the effect of this difference by emphasizing the nuclear surface. Such an effect might permit a test of the isospin purity of nuclei such as  $\text{Ca}^{40}$ . A comparison of the various calcium isotopes might even show (if it occurs) the change from a predominantly protonic to neutronic surface as  $N-Z$  increases. The method would be to observe an isospin *changing* reaction such as  $p + \mathcal{N} \rightarrow \mathcal{N} + N^*(1238)$ . In this case, the target might be a scintillating  $\text{CaF}_2$  crystal, with the appropriate Ca isotope [12]. Estimates of the magnitude of the effect are required to make this a realistic proposal.

In summary, it is possible to calculate high energy coherent nuclear production amplitudes, at least in a crude manner. The nuclear radius  $R$  is sufficient to determine a diffraction pattern with characteristic width  $\Delta q \sim R^{-1}$ , but predictions of absolute magnitude depend on amplitudes for production on a single nucleon, and on the complex refractive index of nuclear matter for the incoming and outgoing waves  $X$  and  $Y$ , as derived from the elastic scattering amplitudes of  $X$  and  $Y$  on single nucleons.

#### 4. MISCELLANY AND CONCLUSIONS

There are at least two types of non-coherent reaction which have interest for high energy physics. One is the use of the momentum of the bound nucleons to provide a higher center of mass energy than is attainable with a hydrogen target — a cheap “storage ring” [13]. A second type is the double reaction inside the nucleus,  $X + N_1 \rightarrow N_1 + Y_1$ ;  $Y_1 + N_2 \rightarrow N_2 + Y_2$ . This could conceivably permit the study of unstable particle scattering (e.g.,  $\rho N$  scattering), but more plausibly it might lead to observations of products  $Y_2$  which are not easily produced in collisions with nucleons of the standard projectiles,  $p, \bar{p}, \pi, K, \bar{K}, e, \bar{e}, \gamma$ .

In summary, aside from the study of nuclear structure [14], elementary particle collisions with nuclei may provide useful information on basic particle-particle interactions. The reader can doubtless add his own applications to those listed here, and we hope he will.

## ACKNOWLEDGEMENTS

We wish to thank H. Bingham, G. Chew, R. Lander, H. Lubatti, L. Wang, and K. Watson for helpful comments.

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# DIFFRACTION SCATTERING OF STRONGLY ABSORBED PARTICLES

T. E. O. ERICSON

*CERN – Geneva*

*(Received June 17, 1965)*

At incident energies above the classical barrier, the elastic scattering on nuclei of nucleons, deuterons, alpha particles and heavy ions is qualitatively classical diffraction scattering on a charged object. Two entirely different phenomenological approaches are used for its description.

The first, and the commonest, approach is that of a complex optical model potential. The real and imaginary parts of the potential have even for spinless particles to be characterized each at least by strength, radius and surface thickness, i.e., by a minimum of six parameters, which may be reduced to four by taking the radii and surface thicknesses to be equal for the two parts. The description is generally successful in its six or more parameter forms. It suffers conceptually from the following points for strongly absorbed particles [1]: the results for large angle scattering seem to depend sensitively on the behaviour of the potential in the surface region at distances for which the matter density is small; the interior properties of the potential are essentially irrelevant. The present potential approaches do not explicitly extract the critical properties of the potential, nor do they systematically minimize the number of parameters [2]. The massive computer calculations needed to obtain the scattering amplitude make the connection between the potential and the amplitude far from transparent and simple.

The second approach tries directly to make an intelligent approximation to the partial wave amplitudes  $\eta_l$  as a function of  $l$ , and to single out those features of  $\eta_l$  on which the scattering amplitude depends crucially. The merit of this approach is the qualitative insight it gives into the properties of the scattering amplitude. Further, it permits the use of a minimal number of parameters which may be in-

creased as the experiments require a more detailed description. This approach enquires at most qualitatively into the actual mechanism of the scattering; it should rather be looked upon as a method to isolate the essential points which must be correctly described by a dynamical model. While this method basically is very simple, it should realistically include Coulomb effects which normally necessitates the use of a computer. The purpose of this article is to show that there exists a model for  $\eta_l$  of a strongly absorbed particle for which the large angle scattering amplitude can be obtained in a very simple closed form with Coulomb effects included. Further, the model is only the simplest of a large class of more general models for the  $\eta_l$  for all of which the large angle scattering amplitude can be obtained to an excellent approximation and in a very simple form.

Consider the partial wave expansion of the scattering amplitude  $f(\theta)$  of a spinless particle:

$$f(\theta) = (2ik)^{-1} \sum_l (2l+1)(\eta_l - 1)P_l(\cos \theta). \quad (1)$$

It is well known that the classical black disc diffraction approximation without Coulomb forces gives an exact closed expression for the amplitude:

$$f_{\text{BD}}(\theta) = \frac{1}{2}i\lambda(L+1) \frac{P_L(\cos \theta) - P_{L+1}(\cos \theta)}{1 - \cos \theta} \simeq i\lambda L(L+1)J_1(L\theta)/L\theta$$

$$\begin{cases} \eta_l = 0 & \text{for } l < L \\ \eta_l = 1 & \text{for } l > L. \end{cases} \quad (2)$$

Here, the cut-off angular momentum  $L$  is to be approximately identified with centrifugal cut-off  $L \simeq kR$ . The corresponding approximation for charged particles is generally referred to as the Blair model [1]. It differs from Eq. (2) by giving all  $l > L$  their correct Coulomb phase shifts, i.e.,

$$\eta_{l>L} = \exp \{2i\sigma_l\} = \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} \simeq (l+\frac{1}{2})^{2i\eta}$$

where  $\eta$  is the ordinary Coulomb parameter:

$$\eta = \frac{Z_1 Z_2 e^2}{\hbar v}. \quad (3)$$

The corresponding scattering amplitude is a very good first approximation to the forward scattering, but it fails badly at large angles [2].

The abrupt rise of  $\eta_l$  from zero at  $l = L$  is obviously an unphysical feature of the black disc model. A more realistic model is a black disc with a grey, partially transparent edge. The approximation is thus to regard the nuclear amplitude in the absence of Coulomb effects to be purely absorptive, i.e.,  $\eta_l$  real, and to interpolate between 0 and 1 by various functions [3, 4]. The partial amplitudes are then given the Coulomb phase corresponding to a point charge. This is a reasonable approximation also in the interior region  $l < L$ , provided this phase varies slowly with  $l$  in this region, since the larger uncertainty will be in the neglect of the nuclear phase. The condition of slow variation is that the classical deflection angle by the Coulomb field for a grazing collision is small, i.e.,  $\theta_C = (L + \frac{1}{2})^{-1} 2\eta < 1$ . The detailed way in which the edge is rounded seems to be of minor importance. Experiments are much better reproduced by the rounded edge distributions than by the sharp cut-off distributions. A particular form of these two-parameter descriptions is the Fermi shape

$$\eta_l = \frac{\exp \{2i\sigma_l\}}{1 + \exp \{(L-l)/a\}} \quad (4)$$

The additional parameter  $a$  should be looked on as a skin thickness. The black disc results as a special case for  $a = 0$ . In order to make a meaningful distinction between the black and the grey disc, we must have  $a \gtrsim \frac{1}{2}$  as otherwise the rise from zero at  $L$  would occur in an interval  $\Delta l < 1$ .

It seems not to have been noticed that Eq. (4) gives a simple closed form for the scattering amplitude at angles  $\pi a \theta - \theta_C > 1$ . Thus, strangely enough, it seems easier to find a closed expression for the grey disc than for the black disc scattering amplitude.

Consider the partial amplitudes  $\eta_l$  to be an analytical function  $\eta(l)$  of  $l$ . The Sommerfeld-Watson transformation can now be applied: the sum is first converted into an integral

$$f(\theta) = \frac{1}{2\pi i} \frac{\pi}{2ik} \int_C \frac{(2l+1)[\eta(l)-1]P_l(-\cos \theta)}{\sin \pi l} dl \quad (5)$$

where the path of integration  $C$  is shown in Fig. 1. The contour  $C$  for a well behaved  $\eta(l)$  is then changed into an integral from  $-\frac{1}{2}-i\infty$  to  $-\frac{1}{2}+i\infty$  plus contributions from the poles  $\alpha_n$  of  $\eta(l)$  which are to the right of  $-\frac{1}{2}$  in the complex  $l$  plane. The residues of these poles are  $\beta_n$ . Thus

$$f(\theta) = (2k)^{-1} \int_{-\infty}^{+\infty} \frac{y[\eta(-\frac{1}{2}+iy)-1]P_{-\frac{1}{2}+iy}}{\cosh \pi y} dy + i\pi(2k)^{-1} \sum_n \frac{\beta_n(2\alpha_n+1)P_{\alpha_n}(-\cos \theta)}{\sin \pi \alpha_n}. \quad (6)$$

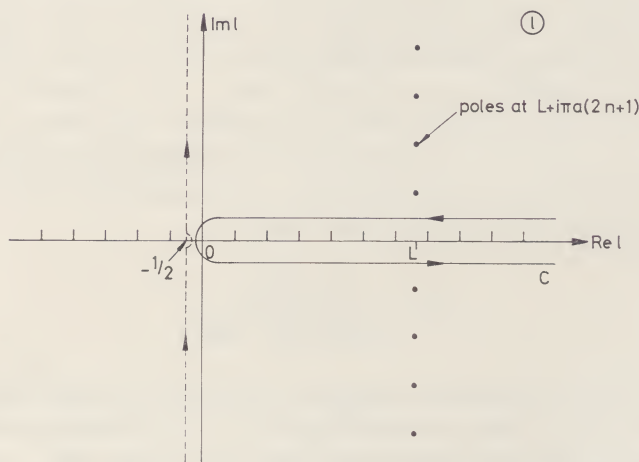


Fig. 1. The paths of integrations in the complex plane of Eqs. (5) and (6) and the positions of the poles of  $\eta(l)$  as given by Eq. (4).

To visualize the importance of the various poles at a scattering angle  $\theta$  we recall that

$$P_l(\cos \theta) \simeq \left( \frac{2}{(l+\frac{1}{2})\pi \sin \theta} \right)^{\frac{1}{2}} \cos \left[ (l+\frac{1}{2})\theta - \frac{1}{4}\pi \right] \quad \text{for } 1/|l| < \theta < \pi - 1/|l| \quad (7)$$

to a very good approximation.

The contribution of a typical pole of Eq. (6) for an angle  $\theta$  will thus be characterized by



$$\begin{aligned}
\frac{P_{\alpha_n}(-\cos \theta)}{\sin \pi \alpha_n} &\simeq i \left( \frac{2}{(\alpha_n + \frac{1}{2})\pi \sin \theta} \right)^{\frac{1}{2}} \times \\
&\times \frac{\exp \{i[(\alpha_n + \frac{1}{2})(\pi - \theta) - \frac{1}{4}\pi]\} + \exp \{-i[(\alpha_n + \frac{1}{2})(\pi - \theta) - \frac{1}{4}\pi]\}}{\exp \{i\pi \alpha_n\} - \exp \{-i\pi \alpha_n\}} \\
&\simeq i \left( \frac{2}{(\alpha_n + \frac{1}{2})\pi \sin \theta} \right)^{\frac{1}{2}} \begin{cases} -\exp \{i[(\alpha_n + \frac{1}{2})\theta - \frac{1}{4}\pi]\} & \text{for } \text{Im } \alpha_n > 0 \\ \exp \{-i[(\alpha_n + \frac{1}{2})\theta - \frac{1}{4}\pi]\} & \text{for } \text{Im } \alpha_n < 0 \end{cases} \\
&\text{and for } |\text{Im } \alpha_n|(\pi - \theta) > 1. \quad (8)
\end{aligned}$$

The amplitude of a pole contribution is thus dominated by  $\exp \{-|\text{Im } \alpha_n|\theta\}$ , i.e., the contributions from poles far from the real axis rapidly vanish when the scattering angle is increased. It is easily seen that this is so also for scattering angles close to  $180^\circ$  for which the approximation (7) is invalid. We may therefore hope that the scattering amplitude at larger angles is dominated only by the one or two poles closest to the real axis and by the integral term in Eq. (6).

We now apply these results to the grey disc approximation (4). The Fermi distribution is an analytic function of  $l$  as required, with poles lying equidistant on a line through  $L$  parallel to the imaginary axis [see Fig. 1]\*. The exact positions and residues of these poles are

$$\begin{aligned}
\alpha_n &= L + i\pi a(2n+1); \\
\beta_n &= a \exp \{2i\sigma(\alpha_n)\} \simeq a \exp \{2i\sigma(|\alpha_n + \frac{1}{2}| - \frac{1}{2})\} e^{-2\eta\varphi_n} \quad (9)
\end{aligned}$$

where  $n$  takes on all integral values from  $-\infty$  to  $+\infty$  and where  $\varphi_n = \arctg \{2\pi a(2n+1)/(2L+1)\}$ . We notice that all the residues are equal in the absence of Coulomb forces. The importance of the  $n^{\text{th}}$  pole is, according to Eqs. (8) and (9), determined by

$$\exp \{-2\eta\varphi_n - \pi a\theta|2n+1|\} \simeq \exp \{-|2n+1|\pi a(\theta \pm \theta_c)\}$$

for  $\pi a < L$ . It is clear that for  $\pi a(\theta - \theta_c) > 1$  only the two poles nearest

\* In order to avoid confusion, we emphasize that the poles of Eq. (4) are not Regge poles. Equation (4) is an approximation to the finite number of  $\eta_l$  which gives important contributions to the amplitude. The use of the poles of Eq. (4) is thus a mathematical trick to get an insight into the dependence of the scattering amplitude on the parameters. There is no physical significance of these poles; neither does one have any right to believe that a good description of the scattering amplitude at moderate angles necessarily implies that the largest angles will be described by one or two poles only.

the real axis can be of importance. Further if in addition  $\pi a \theta_C > 1$  only one of these poles contributes.

The integral term in Eq. (6) can easily be evaluated. For our present purposes it is sufficient to know that it is  $f_C(\theta) \exp\{-L/a\}$  for small and moderate angles ( $f_C(\theta)$  is the point charge Coulomb amplitude); for large angles the integral contributes considerably less. Unless the pole contributions fall to very small value, this term is in practice negligible.

The grey disc scattering amplitude from (4) is thus

$$f_{GD}(\theta > \theta_C + (\pi a)^{-1}) \simeq \frac{1}{2} i \pi a \lambda \exp \{2i\sigma(|\alpha_0 + \frac{1}{2}| - \frac{1}{2})\} \times \\ \times \left[ e^{-2\eta\varphi_0} (2\alpha_0 + 1) \frac{P_{\alpha_0}(-\cos \theta)}{\sin \pi \alpha_0} + e^{2\eta\varphi_0} (2\alpha_0^* + 1) \frac{P_{\alpha_0^*}(-\cos \theta)}{\sin \pi \alpha_0^*} \right]. \quad (10)$$

For pedagogical reasons we will from now on use the approximate form (7) for the  $P_l(\cos \theta)$ . Provided we avoid extreme backward angles larger than  $\pi - (\pi a)^{-1}$  we can write Eq. (10) approximately in the very simple form

$$f_{GD}(\theta_C + (\pi a)^{-1} < \theta < \pi - (\pi a)^{-1}) \simeq \frac{2i\pi a \lambda}{\sqrt{\pi \sin \theta}} \times \\ \times |2\alpha_0 + 1|^{\frac{1}{2}} \exp \{2i\sigma(|\alpha_0 + \frac{1}{2}| - \frac{1}{2})\} \sin [(L + \frac{1}{2})\theta + \frac{1}{2}\varphi_0 - \frac{1}{4}\pi + 2i\eta\varphi_0]. \quad (11)$$

The corresponding differential cross-section is

$$\left( \frac{d\sigma}{d\Omega} \right)_{GD} \simeq \\ \simeq \frac{4\pi \lambda^2 a^2 |2\alpha_0 + 1|}{\sin \theta} \{ \sin^2((L + \frac{1}{2})\theta + \frac{1}{2}\varphi_0 - \frac{1}{4}\pi) + \sinh^2(2\eta\varphi_0) \} e^{-2\pi a \theta} \\ \text{for } \theta_C + (\pi a)^{-1} < \theta < \pi - (\pi a)^{-1}. \quad (12)$$

The characteristic features of the cross-section (12) are the following:

a) the skin thickness  $a$  gives rise to an exponential over-all decrease of the cross-section with angle.

b) Superimposed on this decrease are undamped regular oscillations persisting until the largest angles with an amplitude governed by the Coulomb parameter. Without Coulomb forces these oscillations are very strong. Their amplitude decreases at first rather slowly with increasing Coulomb parameter  $\eta$ , and very rapidly once  $\pi a \theta_C > 1$ .

This reflects that a single pole in Eq. (10) becomes predominant. In the very large angle region beyond  $\pi - (\pi a)^{-1}$  the oscillations recur again and become strong, since the square of the only contributing complex Legendre function oscillates rapidly there. More specifically: in the one-pole case the differential cross-section has a maximum at  $180^\circ$ ; further, the angular distribution in terms of  $\theta' = \pi - \theta$  is given by

$$|P_{a_0}(\cos \theta')|^2 \simeq |J_0(\alpha_0 \theta')|^2 \simeq |J_0((L + \frac{1}{2})\theta')|^2$$

close to  $180^\circ$ , which is diffraction-like.

c) In the special case we discuss here the modification of the cross-section by the Coulomb forces is simply an additive term which decreases exponentially with angle. This implies in particular that a strong Coulomb potential will cause a very considerable enhancement of the cross-section at all large angles, even though these angles are much larger than the classical deflection angle.

The grey disc model we have discussed so far accounts qualitatively, and is from previous computer calculations known to account quantitatively, for many aspects of elastic scattering with strong absorption. It is manifestly unphysical in so far that it entirely neglects the non-absorptive contribution in the purely nuclear amplitude, which corresponds to a non-zero nuclear phase shift. The nuclear phase shift is therefore included in realistic models for  $\eta_i$  [3, 4]. From our point of view there are two specific consequences of the neglect of the nuclear phase shift, namely that the two dominating poles are symmetrically placed with respect to the real axis and that their residues are equal in the absence of Coulomb forces. The introduction of a nuclear phase could therefore:

i) lead to differing residues for the poles. This would show up as an anomalous Coulomb effect in the large angle region, but would lead to no other qualitative changes.

ii) shift the position of the poles so that one will be closer to the real axis than the other. This leads to a qualitatively new effect: the contributions of the poles to the amplitude decrease exponentially at a different rate. After initial interference and oscillations at smaller angles, one of the poles begins to dominate. The amplitude of the oscillations decreases and the cross-sections decrease nearly exponentially

at larger angles. Eventually at the very largest angles oscillations reoccur similarly as in case of a strong Coulomb interaction.

While it is simple to solve a more general pole approximation to the scattering amplitude by the same methods, the advantage of using it is debatable, since even two poles require eight parameters (of which one is an irrelevant phase) for their description, which is far too much freedom. On the other hand, it is reasonable to try a pole description of the expected character of the nuclear phase by fewer parameters. It is clear that the behaviour of the phase is relevant only in the surface region, since for smaller  $l$  there is no amplitude and for larger  $l$  the centrifugal barrier suppresses the nuclear phase. In the surface region we expect the nuclear phase to vary rapidly, due to the peculiar effects associated with the impedance mismatch for a wave passing just above a barrier [5]. It is clear that poles close to the real axis will give rise to rapidly varying phases, and that they thus should be suitable to describe this phenomenon. In addition to the poles, we must require that the general behaviour of the  $\eta(l)$  is unchanged and that unitarity is not violated. This can usually be done in various ways.

A simple way to achieve a three-parameter description of the partial amplitude is simply to displace the nuclear part of Eq. (4) with respect to real axis by introducing a complex cut-off  $L' = L + i\lambda$ .

$$\eta(l) = \frac{\exp \{2i\sigma(l)\}}{1 + \exp\{(L + i\lambda - l)/a\}} \quad \text{with } |\lambda| < \frac{1}{2}\pi a. \quad (13)$$

This form satisfies unitarity and rises from 0 similarly as Eq. (4) with increasing  $l$ . The nuclear phase shift goes rapidly from  $-\arctg\{\lambda/a\}$  to zero for a change of  $l$  of the order of  $a$  for  $l \approx L$ . The two poles closest to the axis are now at  $L + i(\pi a - \lambda)$  and  $L - i(\pi a + \lambda)$ . Consequently one of the poles will dominate for angles  $\theta > \frac{1}{2}|\lambda|^{-1}$ , with a qualitative behaviour of the cross-section as described above. This scattering amplitude and the cross-section can be immediately obtained in a form analogous to Eqs. (1)–(12).

The three-parameter form (13) exhausts in a certain sense those features of the strong absorption  $\eta_l$  which intuitively are of great importance to the elastic scattering: the cut-off in the skin thickness and the rapid phase variation at the surface.

The main feature of  $\eta_l$  that is not describable by the three-para-



meter form is a rapid sign change of the nuclear phase. It is possible to describe this effect by slightly displacing the real part of the two poles with respect to each other. It is seen from Eqs. (8) and (9) that this would give rise to a non-regular oscillatory behaviour of the cross-section, which thus is a qualitatively new effect associated with the phase shift behaviour.

The pole models seem to be capable of describing all the main features of the strong absorption cross-sections and many of its details using very few parameters. Due to their solvable form at large angles, it is immediately possible to see the consequences for the elastic scattering of a change in the behaviour of the complex phase shift.

It is a pleasure to acknowledge profitable discussions with Dr. J. Høgaasen and Dr. T. Høgaasen on the evaluation of the integral term.

This article is dedicated to Professor V. F. Weisskopf, who has contributed so significantly to the clarification of nuclear reaction theory and who has always succeeded in extracting the essential physics out of complicated formalism.

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# PION-NUCLEON SCATTERING AND $SU(4)$ SPIN-ISOSPIN SYMMETRY

M. CINI

*Istituto di Fisica e Scuola di Perfezionamento in Fisica*

*Università di Roma*

*Istituto Nazionale di Fisica Nucleare – Sezione di Roma*

*(Received June 21, 1965)*

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Attention has been called recently [1] on the possibility that a reconsideration of Wigner's  $SU(4)$  theory of supermultiplets [2] might be a useful tool for the study of the properties of non-strange baryons and mesons. However the claim that this theory gives new predictions about the electromagnetic properties of these particles has been questioned [3]. It may therefore be interesting to investigate in some detail to what extent the approach based on the enlargement to the spin space of the symmetry properties of internal degrees of freedom does indeed provide new information in comparison with the conventional dynamical theory. In other words one should ask whether both the assignment of different particles to the same representation of the symmetry group and the relationship between different interaction couplings and amplitudes which follow under such assumptions, might be justified in terms of simpler dynamical properties rather than being postulated as primary and fundamental.

With the purpose of illustrating this point of view and mainly for pedagogical purposes we will present in the following some observations on the relation between the conventional treatment of the old problem of pion-nucleon scattering, and its modern version in terms of the  $SU(4)$  spin-isospin symmetry group. The conclusions, although derived in a very limited context, might be of more general validity, and help to shed some light also on the meaning of  $SU(6)$  [4].

2

The main feature of pion-nucleon scattering at low energies is the

dominance of the  $\frac{3}{2} \frac{3}{2}$  resonant state. Available theories reproduce correctly this feature in the sense that they give a resonant solution in the right state and provide a relation between pion-nucleon coupling constant and width of the resonance which turns out to be in very good agreement with experiment.

The position of the resonance, however cannot be determined, because it depends largely on the unknown behaviour of the amplitude at high energies and has to be regarded to a large extent as an arbitrary parameter [5].

We briefly recall in what follows the main results of a crude but essentially valid treatment [6] of the problem. We start from the dispersion relation for the partial wave  $h_{33} = e^{i\delta_{33}} \sin \delta_{33}$

$$h_{33}(\omega) = \frac{4}{3}f^2 \frac{k^3}{\omega} + \frac{k^3}{\pi} \int_1^\infty \frac{d\omega'}{k'^3} \text{Im } h_{33}(\omega') \left\{ \frac{1}{\omega' - \omega} + \frac{1}{9} \frac{1}{\omega' + \omega} \right\}. \quad (1)$$

With good approximation Eq. (1), as is well known, has a resonant solution of the form

$$h_{33}(\omega) = \frac{\gamma k^3(\omega^*/\omega)}{\omega^* - \omega - i\gamma k^3(\omega^*/\omega)} \equiv \frac{N_{33}(\omega)}{D_{33}(\omega)} \quad (2)$$

where the resonance energy  $\omega^*$  is an undetermined parameter. The partial width  $\gamma$ , on the other hand, is related to the coupling constant  $f$  by

$$\gamma \sim \frac{3}{2}f^2 \quad (3)$$

obtained by imposing that the numerator  $N_{33}(\omega^*)$  at the resonance pole should be given essentially by the contribution of the nucleon pole and the left hand cut as follows:

$$\gamma \sim \frac{4}{3}f^2 + \frac{1}{9}\gamma. \quad (4)$$

Equation (3) is slightly different from the one generally used in the Chew-Low effective range formula  $\gamma = \frac{4}{3}f^2$  obtained by neglecting in Eq. (4) the crossed contribution of the resonance. Equation (3) is more satisfactory because is self consistent, from the point of view of the bootstrap method, with the relation obtained by considering the nucleon as a bound state held together by the exchange of a (33) resonance between pion and nucleon [7].

The solution (2) can now be used to give explicitly the phaseshifts in all the scattering states, provided one neglects in the dispersion integrals the contributions of the non resonant states, and, since the contribution of the resonance is sufficiently peaked near the resonance energy  $\omega^*$ , one makes the replacement

$$\int \frac{\sin^2 \delta_{33}(\omega')}{k'^3} \frac{d\omega'}{\omega' + \omega} \approx \frac{1}{\omega^* + \omega} \int \frac{\sin^2 \delta_{33}(\omega')}{k'^3} d\omega' \approx \frac{\pi\gamma}{\omega^* + \omega}. \quad (5)$$

Furthermore, when  $\omega$  is sufficiently far from  $\omega^*$  one can also make the still cruder approximation

$$\int \frac{\sin^2 \delta_{33}(\omega')}{k'^3} \frac{d\omega'}{\omega' - \omega} \approx \frac{\pi\gamma}{\omega^* - \omega}. \quad (6)$$

Finally one obtains [8]

$$h_{TJ} = \frac{1}{3} \begin{pmatrix} -8 \\ -2 \\ -2 \\ 4 \end{pmatrix} f^2 \frac{k^3}{\omega} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \gamma \frac{k^3}{\omega^* - \omega} + \frac{1}{9} \begin{pmatrix} 16 \\ 4 \\ 4 \\ 1 \end{pmatrix} \gamma \frac{k^3}{\omega^* + \omega}. \quad (7)$$

Equation (7) can be obtained directly by using a different language [9]. One could have started from the knowledge that a nucleon isobar  $\Delta(1238, \frac{3}{2}^+)$  exists and could have tried to obtain a reasonable accurate scattering matrix at low energies by taking Born approximation corresponding to all the known particle and isobar poles with the appropriate residues [10]. In this case, treating the isobar as a stable particle one would start with a fixed source effective interaction Hamiltonian [11] in which the pion field is coupled not only to the nucleon, but also to the isobar, and induces transitions from the former to the latter. Its form will be:

$$H' = \sum_{k\alpha} [V_{\alpha}^{NN}(\mathbf{k}) + V_{\alpha}^{NA}(\mathbf{k}) + V_{\alpha}^{AN}(\mathbf{k}) + V_{\alpha}^{AA}(\mathbf{k})] a_{\alpha}(\mathbf{k}) + \text{h.c.} \quad (8)$$

where, as usual

$$V_{\alpha}^{NN}(\mathbf{k}) = i f \frac{1}{\sqrt{2\omega_k}} N^+ (\boldsymbol{\sigma} \cdot \mathbf{k}) \tau_{\alpha} N \quad (9)$$

while we define



$$V_{\alpha}^{\Delta N}(k) = i \frac{f^*}{\sqrt{3}} \frac{1}{\sqrt{2\omega_k}} \Delta^+(\Sigma \cdot k) \Theta_{\alpha} N. \quad (10)$$

In (9), (10)  $N^+$  is the usual fixed nucleon creation operator with two spin and two isospin components while  $\Delta^+$  is a four spin and four isospin component isobar creation operator. The operators  $\Sigma_i$  ( $i = 1, 2, 3$ ) are rectangular matrices with four rows and two columns operating on the right on the two component nucleon's spinor and on the left on the four component isobar's spinor. The matrices  $\Theta_{\alpha}$  ( $\alpha = 1, 2, 3$ ) have the same form and act on the corresponding isospin components. We need not give explicitly the term  $V_{\alpha}^{\Delta\Delta}$  in (8) because it corresponds to pion emission or absorption by the isobar and does not contribute to pion-nucleon scattering. The elements of the matrices  $\Sigma_i$ ,  $\Theta_{\alpha}$  are easily found with the use of standard Clebsch-Gordan coefficients as follows

$$(\Sigma \cdot k) = \begin{pmatrix} \sqrt{\frac{3}{2}}(k_x - ik_y) & 0 \\ -\sqrt{2}k_z & \frac{1}{\sqrt{2}}(k_x - ik_y) \\ -\frac{1}{\sqrt{2}}(k_x + ik_y) & -\sqrt{2}k_z \\ 0 & -\sqrt{\frac{3}{2}}(k_x + ik_y) \end{pmatrix}. \quad (11)$$

We can use  $\Sigma$  to construct the  $\frac{3}{2}$  projection operator  $P_3$  exactly as with  $\sigma$  we construct the  $\frac{1}{2}$  projection operator  $P_1$ :

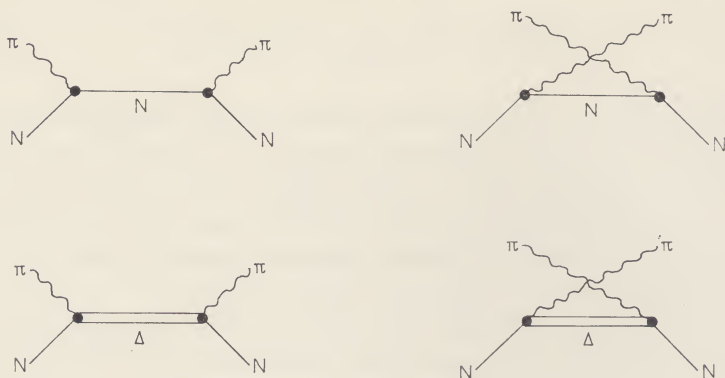
$$\begin{aligned} \frac{1}{3}\sigma_i\sigma_{i'} &= (P_1)_{ii'} \\ \frac{1}{3}\Sigma_i^+\Sigma_{i'} &= (P_3)_{ii'}. \end{aligned} \quad (12)$$

$(P_1)_{ii'}$  and  $(P_3)_{ii'}$  are of course, as usual, two-by-two matrices in the nucleon's spin space.

Crossing gives also

$$\begin{aligned} \frac{1}{3}\sigma_{i'}\sigma_i &= \frac{2}{3}(P_3)_{ii'} - \frac{1}{3}(P_1)_{ii'} \\ \frac{1}{3}\Sigma_{i'}^+\Sigma_i &= \frac{1}{3}(P_3)_{ii'} + \frac{4}{3}(P_1)_{ii'}. \end{aligned} \quad (13)$$

By means of the Hamiltonian (8) if one computes the  $T$ -matrix corresponding to the sum of the four diagrams:



one obtains easily Eq. (7) for the pion-nucleon partial amplitudes  $h_{TJ}$  with  $f^{*2} = \gamma$ . This is consistent with the assumption that a crude, but not unreasonable  $T$  matrix is given by Born approximation, with isobars treated as stable particles, provided a definite relation exists between the pion-nucleon coupling constant  $f$  and the pion-nucleon-isobar coupling constant  $f^*$ , namely

$$f^{*2} = \frac{3}{2} f^2. \quad (14)$$

3

Let us now look at the same problem from the point of view of  $SU(4)$  symmetry. The only baryons in our problem are the nucleon and the isobar and we are therefore led *uniquely* to the 20-dimensional representation as the only one in which all the available states can be accommodated [12]. Since their mass difference is *not* due to their interaction with pions, being an arbitrary parameter, it means that interactions responsible for mass splittings are not important in determining their interaction strengths at low energies: *it is this property which justifies in our opinion the assignment to a unique multiplet of the nucleon and the isobar*. The basis of the 20-dimensional representation is the symmetric tensor of third rank  $B^{\alpha\beta\gamma}$  ( $\alpha, \beta, \gamma = 1, 2, 3, 4$ ).

Its explicit dependence on the  $SU(2) \otimes SU(2)$  spin and isospin indices, with the notation  $\alpha \equiv (i, A)$ ,  $\beta \equiv (j, B)$ ,  $\gamma \equiv (k, C)$  is well known [1]:

$$B^{\alpha\beta\gamma} = \chi^{ijk} d^{ABC} + \frac{1}{3\sqrt{2}} [\epsilon^{ij} \epsilon^{AB} \chi^k b^C + \epsilon^{jk} \epsilon^{BC} \chi^i b^A + \epsilon^{ki} \epsilon^{CA} \chi^j b^B] \quad (15)$$

where  $\chi_i, b^A$  are the usual Pauli spin and isospin spinors while  $d^{ABC}$   $\chi_{ijk}$  (third rank symmetric tensors) are the isospin and spin  $\frac{3}{2}$  spinors respectively. Their relation with the components of the  $N, \Delta$  operators previously introduced is:

$$\begin{cases} \chi^1 b^1 = N(\frac{1}{2}, \frac{1}{2}) & \chi^1 b^2 = N(\frac{1}{2}, -\frac{1}{2}) \\ \chi^2 b^1 = N(-\frac{1}{2}, \frac{1}{2}) & \chi^2 b^2 = N(-\frac{1}{2}, -\frac{1}{2}) \end{cases} \quad (16)$$

$$\begin{cases} \chi^{111} d^{111} = \Delta(\frac{3}{2}, \frac{3}{2}) & \chi^{111} d^{112} = \frac{1}{\sqrt{3}} \Delta(\frac{3}{2}, \frac{1}{2}) \\ \chi^{112} d^{111} = \frac{1}{\sqrt{3}} \Delta(\frac{1}{2}, \frac{3}{2}) & \chi^{111} d^{221} = \frac{1}{\sqrt{3}} \Delta(\frac{3}{2}, -\frac{1}{2}) \\ \chi^{122} d^{111} = \frac{1}{\sqrt{3}} \Delta(-\frac{1}{2}, \frac{3}{2}) & \chi^{111} d^{222} = \Delta(\frac{3}{2}, -\frac{3}{2}) \\ \chi^{222} d^{111} = \Delta(-\frac{3}{2}, \frac{3}{2}) & \text{etc.} \end{cases} \quad (17)$$

Let us now come to pions. Pions are assigned in  $SU(4)$ , together with  $\rho$ 's and  $\omega$  to the 15-dimensional representation as follows

$$M_\beta^\alpha = [\delta_j^i \pi_B^A + \sigma_j^i \cdot \rho_B^A + \sigma_j^i \cdot \omega \delta_B^A] \quad (18)$$

with

$$\pi_1^1 = -\pi_2^2 = \frac{1}{\sqrt{2}} \pi^0 \quad \pi_2^1 = \pi^+ \quad \pi_1^2 = \pi^- \quad (19)$$

and similar relations for  $\rho_B^A$  [13].

To construct the effective baryon-meson vertex one cannot of course couple directly  $B_{\alpha\beta\gamma}^+ B^{\alpha\beta\delta}$  with  $M_\delta^\gamma$  because this would lead to an s-wave pion-nucleon coupling. One has to define instead [15]

$$P_\beta^\alpha = M_\gamma^\alpha Q_\beta^\gamma \quad (20)$$

with

$$Q_\beta^\gamma = (\sigma \cdot k)_j^k \delta_B^C \quad (21)$$

and write the baryon-meson vertex as

$$H_1 = 6f_1 B_{\alpha\beta\gamma}^+ B^{\alpha\beta\delta} P_\delta^\gamma. \quad (22)$$

The effective Hamiltonian  $H_1$  is the  $SU(4)$  reduction of the  $SU(6)$  invariant meson-baryon interaction chosen by Gürsey, Radicati and

Pais [4]. It has however already been remarked [14] that this choice is not at all unique. One could equally well construct another effective Hamiltonian

$$H_2 = 6f_2 B_{\alpha\beta\gamma}^+ B^{\alpha\beta'\gamma'} M_{\beta'}^\beta Q_{\gamma'}^\gamma. \quad (23)$$

Now, if the appropriate spinor reduction is made, both  $H_1$  and  $H_2$  assume the form (8), apart from terms with  $\rho$  and  $\omega$  interactions. The main point is that pure  $f_1$  coupling leads to a pion-baryon effective Hamiltonian with

$$f = \frac{5\sqrt{2}}{3} f_1 \quad f^* = \frac{4}{\sqrt{3}} f_1 \quad (24)$$

namely

$$f^{*2} = \frac{24}{5} f^2. \quad (25)$$

On the other hand pure  $f_2$  coupling would lead to

$$f = -\frac{\sqrt{2}}{3} f_2 \quad f^* = -\frac{2}{\sqrt{3}} f_2 \quad (26)$$

namely

$$f^{*2} = 6f^2. \quad (27)$$

Comparison with (14) shows that pure  $f_1$  coupling approximately satisfies the relation previously found while pure  $f_2$  coupling grossly violates it. It is however, in our opinion, to be stressed that Eq. (14), being closely related to the unitarity condition for the  $T$ -matrix, namely to the statement that the (33) resonance practically saturates the dispersion relation in pion-nucleon scattering, rests on much firmer ground than Eq. (24).

In other words we would rather interpret the  $SU(4)$  effective Hamiltonian  $H_1$  as an *approximate* form of the effective Hamiltonian (8), which, in its turn derives from the approximate solution of the dispersion relation [1]. This is essentially a similar point of view as in the bootstrap philosophy, except that it is much more simple minded and limited in scope. We rely, in fact, only on the conventional old-fashioned dynamics to obtain the relation among coupling constants required for the approximate validity of the symmetry principle. However, from our point of view, the use of the latter does not contain more



physics than the conventional dynamical scheme; although it may well lead, in virtue of the powerfulness of the group theoretical methods used, to predictions which would have been difficult to find with the conventional methods. On the other hand, one should be very careful in trusting too much these results obtained by symmetry arguments alone, if they really are a consequence of approximations introduced in the dynamical equations.

To illustrate this point we go back to our problem. If we consider Eq. (14) as the correct relation between  $f$  and  $f^*$  then we have to use a combination of  $H_1$  and  $H_2$  in order to reproduce it correctly. One can show that

$$f = \frac{\sqrt{2}}{3} (5f_1 - f_2) \quad f^* = \frac{2}{\sqrt{3}} (2f_1 - f_2) \quad (29)$$

and therefore if (14) is imposed one obtains [15]

$$f_1 = -f_2. \quad (30)$$

This shows that predictions obtained by neglecting  $H_2$  altogether may be quite misleading.

We conclude our discussion with a comment on the question of the nucleons magnetic moments, which has received a great deal of attention recently. If our point of view is valid it is clear that  $SU(4)$  has nothing more to say than the conventional theory, namely that the ratio  $\mu_p/\mu_n$  is completely undetermined, since the isovector part of the magnetic moments is connected to the pion nucleon effective interaction while the isoscalar part cannot be related to this quantity. We agree therefore completely with conclusions reached in [3].

#### ACKNOWLEDGEMENT

I am grateful to Prof. F. Calogero for critical reading of the manuscript.

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- 9) Ref. 6) eq. (10.10).
- 10) M. Cini and S. Fubini; Ann. Phys. (N.Y.) **10** (1960) 352.
- 11) By effective Hamiltonian we mean that the coupling constants have their renormalized values.
- 12) There is no 16-dimensional representation in which the isobar can be accommodated separately from the nucleon.
- 13) While the baryon assignment was a consequence of dynamics this assignment does not follow from a knowledge of the dynamical properties of  $\rho$ ,  $\omega$  and pions. One might believe therefore that at this stage the SU(4) classification does introduce something more than conventional theory. This is not true, however, at least for pion-nucleon scattering, because the effect of  $\rho$ 's on this process cannot be described only in terms of an effective interaction between  $B^+$ ,  $B$  and  $M$  but needs also an effective interaction between three  $M$ 's involving an additional parameter.
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# A SEMICLASSICAL APPROACH TO THE PERIPHERAL MODEL

D. AMATI

*CERN - Geneva*

*and*

*Istituto di Fisica dell'Università, Modena*

*(Received June 21, 1965)*

1

At the beginning of 1961, it was gradually becoming evident that the peripheral model was not only a theoretical tool for extrapolating experimental results, but was also able to predict the bulk of experimental information regarding high energy scattering. The fact that the model remained valid even for not too small momentum transfers (by small I mean of the order of the pion mass) was a puzzle to many physicists.

One day in that period, in a discussion around the coffee table, Viki Weisskopf expressed his belief that there must be a way of understanding the peripheral model in terms of a classical picture, which would provide a more intuitive idea of its successes, as well as of its ambiguities and limitations. He asked me if I would try to make up such a classical picture, which would not make use of such terms as pole, Feynman diagram, etc.

In the following weeks, I made some arguments in that direction even though for me the language of singularities and analytic continuations was clearer than that of probabilities and fluxes. Unfortunately, I never had the opportunity of discussing them with Viki as he had in those days a serious automobile accident, which kept him away from us for several months. After that time there was no need for classical arguments because, as always happens in physics, a reasoning becomes classical as one grows accustomed to it, and concepts that were once difficult to grasp become altogether natural by force of habit.

However, when Léon Van Hove offered to me the possibility of

participating in these preludes, I thought it might coincide with the idea of the editors to reproduce that argument about the classical approach to peripherism as an homage to a discussion with Viki which I never had.

Let me state from the very beginning, however, as already mentioned earlier, that I not only believe this classical argument to no longer be really useful, but it is as well probably known to many physicists \*).

2

The semiclassical argument is straightforward and is quite analogous to the Weizsäcker-Williams approach in electrodynamics. Let us take  $\pi N$  scattering and let us fix to the lab. frame. At energies sufficiently high so that the pion associated wavelength is considerably smaller than the proton radius ( $\sim 2$  fm), the pion can investigate the proton structure. The nucleon cloud, except for the innermost part, will be constituted by pions so that – if we leave out the big momentum transfers which explore that “core region” – the  $\pi N$  process should be understandable in terms of the collision of the incoming pion against the pions of the cloud.

Let us call  $\rho(\mathbf{k})d^3\mathbf{k}$  the number of pions in the nucleon cloud with momentum between  $\mathbf{k}$  and  $\mathbf{k}+d\mathbf{k}$ . If the momentum transfer is small, the nucleon recoil will be irrelevant and we shall use this fact in order to calculate  $\rho(\mathbf{k})$  with a fixed source approximation.

Therefore, following our previous discussion, we expect that for small transfer momenta the cross-section on a nucleon will be substantially given by the cross-section of the beam particle on a pion of the cloud, times the probability of finding a pion with the required kinematical conditions in the nucleon cloud.

In particular

$$d\sigma_{\pi N} = \sigma_{\pi\pi}^{\text{total}} \rho(\mathbf{k})d^3\mathbf{k}. \quad (1)$$

Let us now calculate  $\rho(\mathbf{k})$ . In the fixed source theory, the Hamiltonian density is written as

$$H = H_0 + H_1$$

$$H_0 = \sum_{k,i} \omega_k a_{ki}^+ a_{ki} \quad H_1 = \sum_{k,i} V_{ki} a_{ki} + V_{ki}^+ a_{ki}^+ \quad (2)$$

\* One of these, I am sure, is C. Goebel who, by the way, is the pioneer of the peripheral model.



where  $a_{ki}^+$  is the creation operator of a meson of momentum  $\mathbf{k}$  and isospin index  $i$ ,

$$\omega_k = \sqrt{\mathbf{k}^2 + \mu^2} \quad (3)$$

and  $V_{ki}$  is the source function given by

$$V_{ki} = i\sqrt{4\pi} \frac{f^{(0)}}{\mu} \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{\sqrt{2\omega_k}} \tau_i \quad (4)$$

where  $f^{(0)}$  is the unrenormalized coupling constant.

The probability of finding a meson with momentum  $\mathbf{k}$  and charge  $i$  will be given by the expectation value of  $a_{ki}^+ a_{ki}$ , i.e.\*):

$$\rho_i(k) = \frac{\langle N | a_{ki}^+ a_{ki} | N \rangle}{(2\pi)^3} \quad (5)$$

where  $|N\rangle$  represent the physical nucleon state.

Knowing the explicit form of the Hamiltonian [3], it is a simple matter to compute expectation values of creation and annihilation operators [1] and, in particular, the one in Eq. (5). One can easily arrive at

$$\begin{aligned} \rho_i(k) &= \frac{1}{(2\pi)^3} \langle N | V_{ki} \frac{1}{(\omega_k + H)^2} V_{ki}^+ | N \rangle \\ &= \frac{1}{(2\pi)^3} \sum_{N'} \frac{|\langle N' | V_{ki}^+ | N \rangle|^2}{\omega_k^2} + \sum_n \frac{|\langle n | V_{ki}^+ | N \rangle|^2}{(\omega_k + E_n)^2}. \end{aligned} \quad (6)$$

The first term in the right-hand side of Eq. (6) comes from the single nucleon intermediate state ( $\sum_{N'}$  indicates the sum over charge and spin of the nucleon  $N'$ ). In the second term  $|n\rangle$  indicate any possible state besides the nucleon (i.e., nucleon + pions).

From the fact that  $V_{ki}^+ = V_{-k, -i} |\langle n | V_{ki}^+ | N \rangle|^2$  is related to the total cross-section of a pion of momentum  $-\mathbf{k}$  and charge  $-i$  on a nucleon  $N$ , so one can rewrite (6) in the form

$$\rho_i(k) = \frac{f^2}{(2\pi)^2} \frac{k^2}{\omega_k^3} \sum_{N'} |\langle \chi_N, \tau_i \chi_{N'} \rangle|^2 + \frac{k^2}{(2\pi)^4 \omega_k} \int_{\mu}^{\infty} \frac{\sigma_{\pi-i, N}(\omega_p) d\omega_p}{(\omega_k + \omega_p)^2 \sqrt{\omega_p^2 - \mu^2}} \quad (7)$$

where the explicit form of  $V$  given in (4) has been used, together

\* The  $(2\pi)^3$  comes from the number of states per unit normalization volume.

with the definition of renormalized coupling constant  $f$ , in order to write the first term in the right-hand side of (7). We have actually summed over the nucleon spins in order to obtain (7);  $\chi_N$  and  $\chi_{N'}$  are isospin spinors and  $\sigma_{\pi,N}(\omega_p)$  is the total cross-section of a pion of energy  $\omega_p$  on the nucleon  $N$ . From (7) and (1) we can write

$$\begin{aligned} d\sigma_{\pi N}(\omega_k) = & \frac{f^2}{2\pi} \sum_i \sum_{N'} |(\chi_N, \tau_i \chi_{N'})|^2 \frac{k^4}{\omega_k^3} \sigma_{\pi, \pi_i}(\omega') dk d \cos \theta_k + \\ & + \frac{k^4}{\omega_k} \sum_i \frac{\sigma_{\pi, \pi_i}(\omega')}{(2\pi)^3} dk d \cos \theta_k \int_{\mu}^{\infty} \frac{\sigma_{\pi-i, N}(\omega_p) d\omega_p}{(\omega_k + \omega_p)^2 \sqrt{\omega_p^2 - \mu^2}} \end{aligned} \quad (8)$$

where  $\sigma_{\pi, \pi_i}(\omega')$  is the total cross-section of the incident pion over a target pion of charge  $i$  at total  $\pi\pi$  c.m. energy  $\omega'$  given by

$$\omega'^2 = 2E_{\pi}\omega - 2p_{\pi}k \cos \theta_k + 2\mu^2 \quad (9)$$

where  $E_{\pi}$  and  $p_{\pi}$  are respectively the total energy and momentum of the incident pion in the lab. system.

Let us now interpret the result (8). The first term was coming from the nucleon in the sum over intermediate states. This means that the pion of the cloud scatters with the incident pion and leaves the source in its ground state (i.e., leaves the nucleon as a nucleon). Diagrammatically this would be represented by fig. 1.

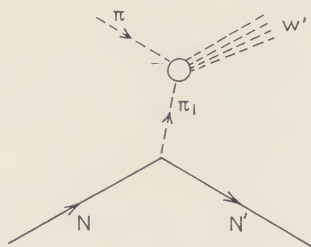


Fig. 1

The second term, instead, represents the situation in which the source is left in an excited state (nucleon + pions) and would represent the situation of fig. 2.

The kinematical variables in (7) can easily be related to the experi-

mental situation:  $k$  and  $-\cos \theta_k$  are simply the momentum and the cos of the angle in the lab. system of the recoil nucleon of fig. 1, or of the recoil system with total energy  $\omega'' \approx m + \omega_p$  in fig. 2. There are the usual limitations between  $k$ ,  $\omega'$  and  $\omega''$  which, however, are not transparent in Eq. (7) due to the fixed source approximation.

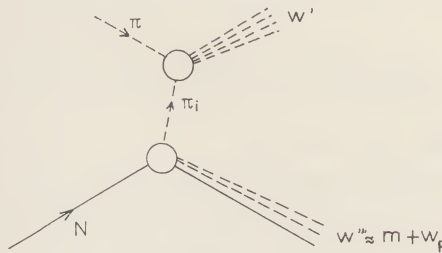


Fig. 2

3

Let us try to compare the results of the semiclassical approach with those of the peripheral model [2], i.e., those obtained on the evaluation of the diagrams of figs. 1 and 2 as Feynman diagrams and some assumptions on the independence of amplitudes on the continuation of masses of scattering particles.

Adding the expressions for the peripheral process of figs. 1 and 2, we obtain

$$\frac{d^2 \sigma_{\pi N}}{d\Delta^2 d\omega'^2} = \frac{1}{2\pi p_\pi^2} \sum_i \frac{\sigma_{\pi, \pi_i}(\omega', -\Delta^2)}{(\Delta^2 + \mu^2)^2} F(p_\pi, \Delta) \left[ \frac{f^2}{\mu^2} \Delta^2 \sum_{N'} |(\chi_{N'} \tau_i \chi_{N'})|^2 + \right. \\ \left. + \frac{1}{8\pi^2} \frac{1}{m^2} \int F(-\Delta, p_N) \sigma_{\pi-i, N}(\omega'', -\Delta^2) d\omega''^2 \right] \quad (10)$$

where  $\Delta$  is the four-momentum of the intermediate pion,  $p_N$  is the one of the target nucleon and  $F(q, q')$  the invariant function which is related to the flux  $f(q, q')$  in a scattering of two particles with momentum  $q$  and  $q'$  by

$$F(q, q') = q_0 q'_0 f(q, q') = \sqrt{(q \cdot q')^2 - m^2 m'^2}. \quad (11)$$

$\sigma(\omega', -\Delta^2)$  are cross-sections for virtual pions of "mass"  $-\Delta^2$ .

In order to compare with our result (8), it is clear that we must restrict to small values of  $k/m$  (in order for a fixed source to be meaningful); then  $\Delta^2 \sim k^2$  and  $\Delta_0 \sim 0$ . By simple kinematical transformations, we arrive at

$$d\sigma_{\pi N} = \frac{2}{\pi} \frac{k^2}{|p_\pi|} \sum_i \sigma_{\pi, \pi_i}(\omega', -k^2) F(p_\pi, \Delta) \left[ k^2 \frac{f^2}{\mu^2} \sum_N |(\chi_{N'}, \tau_i \chi_N)|^2 + \right. \\ \left. + \frac{1}{4\pi^2 m^2} \int F(-\Delta, \rho_N) \sigma_{\pi-i, N}(\omega'', -k^2) \omega'' d\omega'' \right] \quad (12)$$

where, as before,  $\omega_k^2 = k^2 + \mu^2$ .

In order to actually compare Eq. (8) with Eq. (12), we must decide how we treat the dependence of the "mass" of the pion in the cross sections and in the flux factors. Here we encounter a well-known ambiguity in the peripheral calculation which, in general, is solved by replacing the  $-k^2$  by  $\mu^2$  in  $\sigma$  and leaving the actual kinematical value of  $\Delta$  in the flux\*.

We shall not enter into details of the comparison, but just state the situation. Let us discuss first the contribution to the process of fig. 1. Apart from a constant factor 4, the factor  $F$  brings some angle and energy dependence on the peripheral formula which is different from the semiclassical one. On the whole, the angular dependence is not crucial, while the energy dependence is roughly an extra  $k/\omega$  (with respect to the semiclassical one) if  $F$  is evaluated for a pion of mass  $-\Delta^2$  while it is of the order of 1 if  $F$  is evaluated for a pion of mass  $\mu^2$ .

This result is reasonable if we think that in the semi-classical picture the source is emitting and reabsorbing pions with mass  $\mu$  and therefore they bring along this property in the subsequent processes. A similar situation, even though less clear, due to the appearance of two factors  $F$ , happens for the contribution to the process of fig. 2.

For what regards the factor 4 (on the peripheral side), we must note that together with the semiclassical picture we presented there is another one, i.e., the one in which a pion of the cloud of the incident

\* When not spinless resonances (as the  $\rho$  for instance) are produced, the procedure indicated gives different results from the evaluation of Feynman diagrams in which the resonance is treated as a single particle.



pion hits the nucleon target. It is easy to realize that this process is substantially similar to the preceding one and involves no more parameters, i.e., everything is determined from the interaction of the pion field with the pion and nucleon.

In other words, an intermediate pion can as well be assigned to the pion or to the nucleon cloud. This would give a factor of 2 if we sum probabilities and a factor of 4 if we sum probability amplitudes as we know we must do.

4

We have seen that even if the semiclassical approach does not give exactly the same answer as the peripheral one, the results look similar.

This similarity can allow us to discuss the validity of the approximation, as well as other physical processes, on an intuitive basis. Let us discuss first the validity problem.

In order to use the fixed source approximation we supposed that  $k$  was rather smaller than  $m$  even if reasonably bigger than  $\mu$ . But this is a purely kinematical limitation. There is another one which has dynamical origin and is the following: if we would have multiple scattering of our incident particle in the pion cloud of the target our formulae would break down. So, our results are bound to the limitation that the number of pions in the cloud is not much larger than one.

If we limit our processes to intermediate pions with  $k \leq k_{\max}$ , then the number of pions with this property is

$$\mathcal{N} = \sum_i \int_0^{k_{\max}} \rho_i(k) d^3k. \quad (13)$$

Using the explicit form (7), one can see [1] that, with  $f^2 \sim 0.08$ ,  $\mathcal{N} \sim 1$  for  $k_{\max} \sim 5\mu$ .

We see therefore that there is a dynamical reason which can justify the validity of the peripheral model even for momentum transfers quite larger than the pion mass.

The fact that the value of  $f^2$  must enter in the limitation of the peripheral model is also clear on the basis of diagrams. The advantage of the semiclassical approach is to provide a correlation between  $f^2$  and the maximum momentum transfer for which the model can be expected to be valid, through a well-defined parameter  $\mathcal{N}$ .

Let us discuss now other processes that can be easily understood on the basis of the semiclassical picture.

We have seen that both clouds can act as targets for the other particle. But they could also scatter one another; this would give rise to a multiperipheral process [3]. This chain can be continued through the pion of the cloud of the cloud pion, and so on.

Another process that can be clearly visualized is the diffractive behaviour of the quasi-elastic scattering [4]. Indeed, one possible contribution to the first term in (8) is the elastic diffraction scattering of the incident pion on the pion of the cloud. Due to the fact that diffraction is substantially independent of the charge, the scattered (cloud) pion will have the same charge as before the scattering. So that, due to the fact that the cloud pion together with the source (the nucleon) were in a state  $T = \frac{1}{2}$  before the scattering, they will continue to be in such a state after. It is therefore clear that only the  $T = \frac{1}{2}$  resonances can be excited in this process, as it indeed appears to be.

We see from the preceding discussion that even if nowadays the peripheral model does not need an intuitive introduction, the semiclassical approach can allow to understand in a simple pictorial way some related phenomena.

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# TIME'S ARROW AND EXTERNAL PERTURBATIONS

P. MORRISON

*Physics Department, Massachusetts Institute of Technology*

*Cambridge, Massachusetts*

(Received, June 28, 1965)

Willard Gibbs wrote:[1]

“Let us imagine a cylindrical mass of [continuous] liquid of which one sector of  $90^\circ$  is black and the rest white. Let it have a motion of rotation about the axis of the cylinder in which the angular velocity is a function of the distance from the axis. In the course of time the black and white parts would become drawn out into thin ribbons . . . wound spirally about the axis. The thickness of these ribbons would diminish without limit, and the liquid would therefore tend toward a state of perfect mixture of the black and white portions. That is, in any given element of space, the proportion of the black and white would approach 1 : 3 as a limit. Yet after any finite time, the total volume would be divided into two parts, one of which would consist of the white liquid exclusively, and the other of the black exclusively.”

It is from this *anschaulich* argument of Gibbs that the notion of coarse-graining in statistical mechanics can be held to flow. For it is now clear, as he himself puts it, that the uniformity of equilibrium, which is the result of the stirring in the liquid analogy, is conditional; given any degree of stirring, I can find full non-uniformity if I look very closely. But given any method of defining density (say, in phase space) with a finite cut-off to the information sought, any averaging or “coarse-graining”, and the measurement of uniformity becomes certain. While there is a large and sophisticated literature on this problem\* (which it were folly to claim to know) it remains probably the case that something equivalent to Gibb's coarse-graining process, whether *Stosszahlansatz* or random phase approximation [2], is an essential feature of all studies of the approach to equilibrium, of the

\* *Note added in proof:* Very similar ideas have indeed been published by a number of authors! e.g., J. M. Blatt, *Progr. Theoret. Phys.* **22** (1959) 745.

arrow of time. There is a subjective element to this procedure which is a little disquieting. Are the subtle correlations still present in equilibrium, speaking strictly classically and in ideal cases, or are they not? Is the arrow of time then only an illusion? It is the purpose of this note to answer stoutly that the arrow is real, that is, not subjective, that it is not essentially cosmological, that it arises from an inescapable feature of all physical theory.

Let me begin with a concrete analogue [3]. Across the wall of my office there stretches five meters of computer output. A few hundred small rigid spheres, packed pretty closely into a flat box, have been followed through a couple of thousand collisions. To begin they are arranged in a regular square lattice. Then each ball is given two random velocity components  $v_{x_i}(0)$ ,  $v_{y_i}(0)$ , though all move at the same speed, each starting to move from its lattice position. The collisions go on, and after what amounts to a few collisions per ball, the lattice has been stirred into randomness. The computer prints out "snapshots" of the configuration at our will. At a certain time  $t_R$  the motion is stopped, and the velocity components of each ball reflected, with  $-v_{x_i}(t_R)$  for  $v_{x_i}(t_R)$  and  $-v_{y_i}(t_R)$  for  $v_{y_i}(t_R)$ . Now the motion retraces its wildly complicated path, and after the right number of collisions, plus a collisionless interval to retrace that precise time  $t = 0$  before the first collision, the regular lattice has been marvelously restored. *But the reversibility is not certain.* It is dependent upon a knowing programmer, for the inescapable round-off errors coming from solving the equations of motion with only finite digital accuracy, in the field of rationals, so to speak, will always oppose reversibility, and often leaves the array with the same sort of chaos it had when the reversed motions began at  $t_R$ . Thus the computer has done the equivalent of what coarse-graining can do; it has introduced a subtle sort of noise, arising from finite knowledge, into the classical equations which assert infinite precision, but only in an unattainable analytical calculation. My main point is to add that every classical statement of the laws of motion of any system necessarily leaves out a small physical perturbation, some  $\delta H$ , which cannot in principle be included for finite systems, and which in fact is always amply large enough to prevent a complete retracing.

The argument is elementary, and at bottom not new. It is an inver-



sion of that of Poincaré, who long ago put it that probability itself could be regarded as an illusion, in that the roulette wheel could in principle be taken as purely causal. It is only that the prediction of *rouge* is extraordinarily unstable to error of initial data! By extension, that prediction is also unstable to external perturbations. Therefore, a causal universe, classically without probability at all, becomes a statistical one whenever we consider systems in partial isolation from their context. For then the neglected interactions disturb the predictions of mechanics, and prevent us, say, from unstirring Gibbs's milk and ink. Whenever we choose to place the system boundaries, *something* remains outside which, in sufficient time and for suitably complex systems, will wreck the extraordinarily delicate correlations of position with velocity upon which reversibility, for example, depends. A simple estimate of the degree of sensitivity to external perturbations is the end part of our story. Note that only one system, the whole universe, could possibly exist without any external unknown perturbations. Any theory of a system less complete must allow their presence in some degree. If that degree is adequate, the system becomes irreversible, in spite of the reversibility of dynamics. The whole point is that the intricacy of the ribbons of black threading the phase space rapidly becomes so great for any system of many particles that even dynamically negligible, unshieldable, gravitational perturbations are competent to mix up the pattern. Of course time-reversing both system and perturbation would always work. But that means enlarging the system. For now the perturbation needs to be known, and must become part of the system. Still there remains some other disturbance outside. Only the whole universe can then escape, as it ought to escape, the requirements of the *Stosszahlansatz*.

Consider a system of many particles, say with  $f$  coordinates in phase space. It is located in the neighborhood of some mean  $\bar{p}$  and  $\bar{q}$ , with a range in each:  $\Delta q$ , which represents the edge of its containing volume in coordinate space, and  $\Delta p$ , a measure of the r.m.s. momentum spread as well. It is enough to consider what collisions do to the  $p$  coordinates of the representative point in the hyperspace. At each effective collision,  $p$  coordinates move by an amount roughly equal to the typical measure of  $p$  spread, say  $\Delta p$ . After a time  $T = N\tau_{\text{coll}}$ , where  $N$  is the number of collisions per particle and  $\tau_{\text{coll}}$  a

mean collision time, the representative point has made a wildly complex path in the hyperspace. We may estimate the projected distance between successive crossings of some typical value of one  $p$  coordinate as  $\Delta p/N^m$ , where the power  $m$  of  $N$  may correspond to some sort of random walk. (Whether it is 1 or  $\frac{1}{2}$ , or any small number, makes no difference to our argument.) Now the volume of a typical little grain of momentum space which is missed by the trajectory is about  $(\Delta p/N^m)^f$ . If during a time  $T$  a neglected external force shifts the  $\bar{p}$  value by an amount  $\delta p$ , the volume of momentum space held tangent between new and old trajectory amounts to about  $(\delta p) \times (p^{(f/2-1)})$ . When the empty volume is about equal to the unforeseen volume change, we may expect an error in the trajectory which makes it entirely different from the prediction, at the scale required for prediction. Reversibility, for example, would be lost; only quasi-ergodic predictions would be secure. But this means  $\delta p \sim \Delta p/Nf'$  ( $f' \sim kf$ ,  $k$  is a number of order unity), and then  $T_{\text{mixing}} \approx \tau_{\text{coll}}(\Delta p/\delta p)^{1/f'}$ .

The true solution is so filigreed and braided that the slightest external effect soon shifts it by an amount characteristic of its own scale of detail. One may estimate that a gravitational force exerted by a falling apple a kilometer away over an arc of ten centimeters is ample to mix up the trajectory of a mole of normal gas, in a time of milliseconds. Admittedly this has been a wildly crude estimate, but I do not believe it is in substantial error. For a less complex system, the perturbation becomes of dwindling effectiveness; the solar system cannot be treated as reversible in the presence of galactic forces, but the earth-moon system is easily managed to high accuracy. A few molecules would work equally well simply held in a box.

Gibbs and many followers have emphasized the importance of a *large* strongly-coupled thermostat system in defining the canonical distribution. It seems to me that the least degree of coupling to well-defined dynamical systems is enough to justify statistical mechanics, not with respect to such gross matters as energy relaxation times, but surely to such subtleties as reversibility. Time's arrow is then the necessary consequence of the fact that no physical theory except perhaps the final one can describe the whole of the universe. It seems also clear that the arrow of time in the sense here described

would remain the same for the man who dwells in a contracting, rather than an expanding universe, provided he can once set up, perhaps in some super air-raid shelter, physical systems of the sort we know, temporarily free from large energy inputs out of space. Behind his heaviest shields, gases will leak out of valves irreversibly (unless he pours in the free energy) as they are moved to do by tiny mixing forces out of the external world, however it behaves in the large.

Surely there are other and deeper answers to the problems here touched in an elementary way. But it is worthwhile to try to talk even of these weighty matters in simply physical language, with order of magnitude estimates. There is pleasure and instruction both in such a method. That is what I have learned, however imperfectly, watching with delight the master of the style, V. F. Weisskopf.

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